A Toolkit for Changing Elasticity of Substitution Production Functions

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March 1, 2009

Abstract

This note presents a toolkit for developing production functions with two input factors which have over a freely chosen number of intervals for the input factor intensities different pre-specified elasticities of substitution. The resulting production function can thus account for changing elasticities of substitution during the development of the factor intensity.

Keywords: Elasticity of substitution, Production function

JEL Classification Number: E23, O11, D24

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1 Introduction

The elasticity of substitution between production factors is an important economic parameter. In the case of capital and labor it influences growth, the distribution of income and transitional dynamics towards a balanced growth path (see Karagiannis et al. 2005 and the references therein).

This notes provides a toolkit for building changing elasticity of substitution production functions. For these functions free chosen elasticities of substitution can be attributed to as many discrete factor intensity intervals as is desired. The constructed production function can be a tool e.g. in development studies analyzing the growth behavior of economies[1].

This note builds on the idea of normalizing CES production functions put forward in de La Grandville (1989) and discussed further in Klump and de La Grandville (2000) and Klump and Preissler (2000). Further it makes use of a "functional normalization" first used in Jones (2003) and then generalized in Antony (2009a,b). This approach is distinct from existing ways of modeling a changing elasticity of substitution through the use of variable elasticity of substitution (VES) production functions. First, VES production functions always imply a certain parameterized path of the development of the elasticity which might not be in accordance with the idea of the researcher. Second, VES production functions have, to the best of the author's knowledge, always the property that the elasticity of substitution is either always below or always above one. Crossing this line is thus not possible. Duffy and Papageorgiou (2000) report an elasticity of substitution between capital and labor below one for less developed countries and above one for well developed countries. Considering this type of development with one VES production function is therefore not possible. For such cases the toolkit below might be an alternative.

The next section gives the theoretical background of normalizing CES functions, derives the toolkit and gives an illustrative example. The last section briefly con-

[1] Again referring to the elasticity of substitution between capital and labor, it has been put forward already by Arrow et al. (1969) that this elasticity is changing during the development process which was empirically supported in Duffy and Papageorgiou (2000).

cludes.

2 The Toolkit

2.1 Theoretical Background

In Antony (2009a,b) the following production function is proposed[2]

$$y = f(k) = (\alpha k_b^{\nu} + (1-\alpha))^{\frac{1}{\nu}-\frac{1}{\rho}} \left(\alpha k_b^{\nu} \left(\frac{k}{k_b}\right)^{\rho} + (1-\alpha)\right)^{\frac{1}{\rho}}, \tag{1}$$

where y is output and k is the input factor intensity, e.g. the capital intensity $k = \frac{K}{L}$ where K is capital and L is labor input and y would output per worker. In general $y = \frac{Y}{X_2}$ and $k = \frac{X_1}{X_2}$ with Y the total output and X_1 and X_2 the quantities of production factor one and two. $0 < \alpha < 1$ is a distributional parameter and k_b is some baseline value of the production factor intensity. Taking a closer look at this function it becomes clear that the elasticity of substitution between X_1 and X_2 is $\sigma = \frac{1}{1-\rho}$ for $k \neq k_b$ and $\tilde{\sigma} = \frac{1}{1-\nu}$ for $k = k_b$. More explicitly, the production function (1) normalizes a CES function with elasticity of substitution σ at k_b on a CES function with elasticity $\tilde{\sigma}$.

2.2 The General Production Function

The economic relevance of the production function (1) could be questioned because the change in the elasticity of substitution just happens at one particular value for k. The probability mass for an economy hitting this particular point might be zero. But the production function above can easily be modified to allow for a different behavior of the elasticity of substitution. Consider the modification

$$y = f(k) = (\alpha \max(k_b, k)^{\nu} + (1-\alpha))^{\frac{1}{\nu}-\frac{1}{\rho}} \left(\alpha \max(k_b, k)^{\nu-\rho} k^{\rho} + (1-\alpha)\right)^{\frac{1}{\rho}}, \tag{2}$$

This production function has now the properties that it has an an elasticity of substitution σ for $k < k_b$ and $\tilde{\sigma}$ for $k \geq k_b$.

[2]The special case for $\nu = 0$ was proposed in Jones (2003).

Following this, one can easily extend the function according to

$$
\begin{aligned}
f(k) \quad = \quad & (\alpha \max(k_1,k)^{\rho_1} + (1-\alpha))^{\frac{1}{\rho_1}-\frac{1}{\rho_2}} \\
& \times \left(\alpha \max(k_1,k)^{\rho_1-\rho_2} \max(k_2,k)^{\rho_2} + (1-\alpha)\right)^{\frac{1}{\rho_2}-\frac{1}{\rho_3}} \\
& \times \left(\alpha \max(k_1,k)^{\rho_1-\rho_2} \max(k_2,k)^{\rho_2-\rho_3} \max(k_3,k)^{\rho_3} + (1-\alpha)\right)^{\frac{1}{\rho_3}-\frac{1}{\rho_4}} \\
& \vdots \\
& \times \left(\alpha \max(k_1,k)^{\rho_1-\rho_2} \max(k_2,k)^{\rho_2-\rho_3} \times \cdots \right. \\
& \qquad \left. \cdots \times \max(k_{n-1},k)^{\rho_{n-1}-\rho_n} k^{\rho_n} + (1-\alpha)\right)^{\frac{1}{\rho_n}}
\end{aligned}
$$

A more compact way of defining the production function, although less intuitive when looking at, is

$$
\begin{aligned}
f(k) \quad = \quad & \Pi_{i=2}^{n-1} \left[\left(\alpha \left(\Pi_{j=1}^{i} \max(k_j,k)^{\rho_j-\rho_{j+1}}\right) + (1-\alpha)\right)^{\frac{1}{\rho_i}-\frac{1}{\rho_{i+1}}} \right] \\
& \times \left(\alpha \left(\Pi_{j=1}^{n-1} \max(k_j,k)^{\rho_j-\rho_{j+1}}\right) k^{\rho_n} + (1-\alpha)\right)^{\frac{1}{\rho_n}} \qquad (3)
\end{aligned}
$$

which finally gives the changing elasticity of substitution production function. The elasticity of substitution is $\sigma_i = \frac{1}{1-\rho_i}$ between the values $k_i < k_{i-1}$ for the input factor intensity. For values larger than k_1 the elasticity is $\sigma_1 = \frac{1}{1-\rho_1}$ and for values smaller than k_n it is $\sigma_n = \frac{1}{1-\rho_n}$.

2.3 Properties of the Production Function

The production function (3) is a spline function composed of different CES functions. At the points k_i where the different functions are linked together the function is continuous and differentiable. Thus, through the approach of normalizing the CES functions subsequently at each other it is guaranteed that they share the same value as well as the same slope.

Looking at the derivatives of the function (3) reveals that the first derivative with respect to k is always positive and the second derivative is always negative. This property stems directly from the fact that (3) consists just of different standard CES functions. The first derivative is continuous in k but not differentiable at the points

k_i. Hence, the second derivative shows jumps at the points k_i. This is, of course, just the reflection of the changing elasticity of substitution.

The limiting properties are easily derived. For $k \to \infty$ the function (3) has the same properties as a standard CES function with elasticity of substitution $\sigma_1 = \frac{1}{1-\rho_1}$. For $k \to 0$ it has consequently the limiting property of a CES function with elasticity $\sigma_n = \frac{1}{1-\rho_n}$.

2.4 An Example

Jones (2003) proposes a CES function normalized onto a Cobb Douglas production function at point k^*

$$y = f(k) = (k^*)^\alpha \left(\alpha \left(\frac{k}{k^*} \right)^\rho + (1-\alpha) \right)^{\frac{1}{\rho}}$$

where he refers to k^* as the "appropriate" value of the capital intensity and $\rho < 0$. If the economy is in this appropriate position it enjoys the higher unit elasticity compared to the lower elasticity of substitution $\sigma = \frac{1}{1-\rho}$ anywhere else. He further finds support for such a production function when looking at aggregate and sectoral data on the development of the capital share in production.

If one wants to use this production function e.g. in a numerical analysis the difficulty arises that the probability that the economy exactly hits k^* is almost zero. However, the idea of an "appropriate value" of the capital intensity is very appealing but might be replaced with an "appropriate" region for the capital intensity. This can readily be achieved through the production function (3). Using two linking points $k_1 > k_2$ and an elasticity of substitution of $\sigma = \frac{1}{1-\rho}$ for $k > k_1$ and $k < k_2$ and $\sigma = 1$ for $k_1 \geq k \geq k_2$ and applying L'Hospital's rule gives

$$\begin{aligned} y = f(k) &= \exp\left(\frac{\alpha \max(k_1,k)^\rho}{\alpha \max(k_1,k)^\rho + (1-\alpha)} \ln\left(\frac{\max(k_2,k)}{\max(k_1,k)} \right) \right) \times \\ &\times \left(\alpha \frac{\max(k_1,k)}{\max(k_2,k)})^\rho k^\rho + (1-\alpha) \right)^{\frac{1}{\rho}} \end{aligned}$$

which could serve as a tool in a numerical analysis following the arguments in Jones (2003). During the Cobb Douglas phase the output elasticity of production is given by $\alpha k_1^{\rho}/(\alpha k_1^{\rho} + (1-\alpha))$ and the elasticity of substitution is one for $k_1 > k > k_2$. Anywhere else the elasticity of substitution is $\frac{1}{1-\rho}$.

3 Conclusion

This note presented a changing elasticity of substitution production function where arbitrarily many intervals for the input factor intensity with corresponding elasticities of substitution can be specified. The resulting is spline production function continuously and differentiable linking standard CES functions.

The resulting production function could be a valuable tool e.g. for numerical analysis' in various development studies where the researcher wants to take account of an changing elasticity of substitution between input factors. A straightforward candidate would be the elasticity of substitution between capital and labor.

An advantage of the presented production function over existing variable elasticity of substitution production functions is that the development of the elasticity is not subject to certain patterns implied by the chosen VES function. The approach is flexible as a grid as fine as possible over values for the input factor intensities and elasticities as desired can be chosen.

4 Literature

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