

# How Changing Prudence and Risk Aversion Affect Optimal Saving

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## Abstract

We show how optimal saving in a two-period model is affected when prudence and risk aversion of the underlying utility function change. Increasing prudence alone will induce higher savings only if, for certain combinations of the interest rate and the pure time discount rate, there is distributional neutrality between the two periods. Otherwise, changes of risk aversion that affect the distribution between the periods must also be taken into account.

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# 1 Introduction

A famous result in expected utility theory states that a mean preserving spread of risky exogenous future wealth leads to higher savings if the third derivative of the investor's von Neumann-Morgenstern utility function is positive (see Leland, 1968, Sandmo, 1970, and Drèze and Modigliani, 1972). Utility functions with this property thus reflect a specific precautionary savings motive and accordingly have been coined as “prudent” (Kimball, 1990). Just like different utility functions may show different degrees of risk aversion as indicated by the Arrow-Pratt measure, they may in a quite analogous way also show different degrees of absolute and relative prudence (see also Kimball, 1990, and the exposition in Gollier, 2001). Whereas in some cases a globally higher degree of prudence will increase savings, this assertion is not generally true (see, e.g., Menegatti, 2001, 2007, and Hau, 2002). In this paper we further explore, in the framework of the standard two period model with identical utility functions in both periods, how a higher degree of prudence affects the optimal level of savings. The findings of our analysis are ambiguous: If, through adequate combination of the exogenous interest rate and the pure time discount rate, some equal treatment of the two periods is ensured, higher prudence will induce higher savings. In other cases, the replacement of the utility functions typically has impacts on the distribution of consumption over time such that, in addition, changes of risk aversion have to be taken into account. If risk is low or the interest rate is high, the partial effect brought about by a change of risk aversion will dominate, and the change of prudence becomes irrelevant. Moreover, it can be shown that in the more general case with different utility functions in both periods, it cannot a priori be expected that criteria based only on changes in prudence and risk aversion will generate clear-cut effects on savings behavior.

## 2 The Model

Consider the standard optimal savings model under uncertainty when there are two periods, which we synonymously interpret as two subsequent generations.<sup>1</sup> We first assume that the utility function is the same in the two periods 0 and 1, such that the objective function, i.e. the social welfare function in the intergenerational case, is

$$u(w_0 - s) + \beta E u(\tilde{w}_1 + \rho s). \tag{1}$$

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<sup>1</sup>With this interpretation, our results also have some relevance for the problem of intergenerational distribution which is an important issue, e.g., in the current debate on global warming (see, e.g., Stern, 2006).

Here,  $w_0$  denotes the given certain wealth in the first period,  $\tilde{w}_1$  is the uncertain wealth in the second period and  $s$  is the endogenous amount of savings such that the (safe) consumption in the earlier period is  $c_0 = w_0 - s$  and (risky) consumption in the latter period is  $c_1 = \tilde{w}_1 + \rho s$ . The von Neumann-Morgenstern utility function  $u(c_i)$  (with  $i = 1, 2$ ) is assumed to be defined on  $\mathbb{R}^+$  and to be three times continuously differentiable with  $u'(c_i) > 0$ ,  $u''(c_i) < 0$ , and  $u'''(c_i) > 0$ , i.e. it is strictly monotonically increasing in consumption  $c$ , strictly concave, and prudent.

The marginal rate of transformation  $\rho$  between consumption in period 0 and 1 and the pure time discount factor  $\beta$  are exogenously given by  $\rho = 1 + r$  and  $\beta = 1/(1 + \delta)$  where  $r$  is the interest rate and  $\delta$  is the pure rate of time discount. We assume that maximizing (1) with respect to  $s$  yields an interior solution  $s_u^*$ , which is characterized by the first order condition

$$u'(w_0 - s_u^*) = \beta \rho E u'(\tilde{w}_1 + \rho s_u^*). \quad (2)$$

An important role in our analysis is played by the “precautionary equivalent wealth level”  $\hat{w}_1 = \hat{w}_1(\rho s_u^*, u, \tilde{w}_1)$ , which is defined as the certainty-equivalent of the wealth distribution under optimal savings  $s_u^*$  in period 1 when  $-u'(c)$  is taken to be the utility function. Thus,

$$u'(\hat{w}_1) = E u'(\tilde{w}_1 + \rho s_u^*). \quad (3)$$

In general, the precautionary equivalent wealth level  $\hat{w}_1$  is related to the well-known precautionary equivalent premium  $\psi$  via  $\hat{w}_1 = E\tilde{w}_1 + \rho s_u^* - \psi(\rho s_u^*, u, \tilde{w}_1)$  (see Kimball, 1990, and Gollier, 2001, 128).

The relation between  $\hat{w}_1$  and consumption  $w_0 - s_u^*$  in period 0 then crucially depends on the size of  $\beta\rho$ . In particular,

$$\beta\rho < 1 \quad (= 1, > 1) \quad \Rightarrow \quad \frac{\hat{w}_1}{w_0 - s_u^*} < 1 \quad (= 1, > 1). \quad (4)$$

This assertion follows as (2) and (3) imply

$$u'(\hat{w}_1) = \frac{u'(w_0 - s_u^*)}{\beta\rho} \quad (5)$$

and  $u(c)$  is strictly concave.

We now analyze how optimal savings will change if the utility function  $u(c)$  is substituted by another utility function  $v(c)$ .

### 3 The Results

We assume that the new utility function  $v(c)$  has the same properties as the original utility function  $u(c)$ , i.e. that it is three times differentiable with  $v'(c) > 0$ ,  $v''(c) < 0$ , and  $v'''(c) > 0$ . Furthermore,  $v(c)$  is supposed to be more prudent than  $u(c)$  according to the definition of Kimball (1990), i.e.

$$-\frac{v'''(c)}{v''(v)} > -\frac{u'''(c)}{u''(c)} \quad (6)$$

holds for all consumption levels  $c > 0$ . Hence, if  $v(c)$  is more prudent than  $u(c)$  according to (6), the utility function  $-v'(c)$  is more risk averse than the utility function  $-u'(c)$ . Together with the identity in (3), a standard result concerning changes of Arrow-Pratt risk aversion (see, e.g., Gollier, 2001, 21) then implies

$$v'(\hat{w}_1) < Ev'(\tilde{w}_1 + \rho s_u^*). \quad (7)$$

This result can be used to show that in specific cases higher prudence will induce higher savings.

**Proposition 1.** *If  $\beta\rho$  is sufficiently close to 1, more prudence implies higher savings.*

*Proof.* We first consider the case  $\beta\rho = 1$ . Then,  $\hat{w}_1 = w_0 - s_u^*$  from (4) such that (7) gives

$$v'(w_0 - s_u^*) < Ev'(\tilde{w}_1 + \rho s_u^*). \quad (8)$$

Starting from (8) with  $s = s_u^*$ , it is a straightforward implication of the concavity of  $v(c)$  that  $s$  has to be increased to restore equality, i.e. to get

$$v'(w_0 - s_v^*) = Ev'(\tilde{w}_1 + \rho s_v^*) \quad (9)$$

as the first order condition for optimal savings  $s_v^*$  with the new utility function  $v(c)$ . Therefore,  $s_v^* > s_u^*$  holds in the case  $\beta\rho = 1$  and then, from continuity, also if  $\beta\rho$  is sufficiently close to 1.  $\square$

In general, however, higher prudence alone is not sufficient to provide unambiguous results on an increase in optimal savings. Rather, additional assumptions on an accompanying change of risk aversion are required. We then have two results on the change of optimal savings depending on whether  $\beta\rho < 1$  or  $\beta\rho > 1$ .

**Proposition 2.** *If  $\beta\rho < 1$ , higher prudence combined with higher risk aversion implies higher savings.*

*Proof.* If  $v(c)$  is globally more risk averse according to Arrow-Pratt's standard definition, i.e.  $-\frac{v''(c)}{v'(c)} > -\frac{u''(c)}{u'(c)}$  holds for all  $c > 0$ , the ratio of marginal utilities  $\frac{v'(c)}{u'(c)}$  is decreasing in  $c$ . Since, in the case  $\beta\rho < 1$ , (4) gives  $\hat{w}_1 < w_0 - s_u^*$ , then

$$\frac{v'(w_0 - s_u^*)}{v'(\hat{w}_1)} < \frac{u'(w_0 - s_u^*)}{u'(\hat{w}_1)} = \beta\rho. \quad (10)$$

From (10) and (7), i.e. higher prudence of  $v(c)$ , we get

$$v'(w_0 - s_u^*) < \beta\rho E v'(\tilde{w}_1 + \rho s_u^*). \quad (11)$$

A similar reasoning as at the end of the proof of Proposition 1 then shows  $s_v^* > s_u^*$ .  $\square$

Quite analogously, a result for the case  $\beta\rho > 1$  can be obtained.

**Proposition 3.** *If  $\beta\rho > 1$ , higher prudence combined with lower risk aversion implies higher savings.*

*Proof.* If  $v(c)$  has a lower risk aversion than  $u(c)$ ,  $\frac{v'(c)}{u'(c)}$  is increasing in  $c$ . Since in the case  $\beta\rho > 1$  we have  $\hat{w}_1 > w_0 - s_u^*$ , condition (10) again holds. The proof then continues just like in the case of Proposition 2.  $\square$

We now want to provide some intuitive explanation for these results, which should make it more transparent why savings behavior depends both on prudence and on risk aversion.

## 4 The Interaction of Changes in Prudence and Risk Aversion: An Interpretation

For an interpretation of the results derived in the previous section, we start with the case  $\beta\rho = 1$  in which  $\beta$  and  $\rho$  balance each other. Under the standard assumption that the economy is productive, i.e.  $\rho > 1$  holds, this advantage for the later generation is compensated by a positive pure time discount rate  $\delta > 0$ , i.e.  $\beta < 1$ , so as to avoid an unequal outcome and thus to ensure distributional neutrality. This is a classical justification for pure time preference that dates back to Böhm-Bawerk (1883)(see also, e.g., Arrow, 1999, and – clearly expressed but quite unnoticed – Rawls, 1972, 297-298). How smoothing of consumption across the two generations is brought about by  $\beta\rho = 1$  is particularly obvious in the special case when there is no wealth risk in the later period, i.e. if  $\tilde{w}_1$  is non random.

In this situation,  $\beta\rho = 1$  implies equal consumption levels for both generations. In the case where  $\tilde{w}_1$  is a random variable, the distributional balance between the two generations manifests itself in the identity between consumption in period 0 and the size of the precautionary equivalent wealth level. Then, as described by Proposition 1, the savings level is only affected by changes in prudence since effects on intergenerational distribution are canceled out.

If, however,  $\beta\rho \neq 1$ , things look quite different because in this case, a change of the utility function not only exerts an influence on precautionary savings, but also on the distribution of consumption across generations. First, consider the case  $\beta\rho < 1$  in which the future generation is disadvantaged through a discount rate  $\delta$  that is higher than the interest rate  $r$ , i.e.  $\beta$  is smaller than  $\rho$ . In the benchmark case without wealth risk, the future generation then would have a lower level of consumption than the present generation. With uncertainty in wealth  $\tilde{w}_1$  in period 1, the precautionary equivalent wealth level is lower than consumption in period 1, i.e.  $\hat{w}_1 < w_0 - \rho s_u^*$ . Now, higher prudence still induces higher saving via the precautionary motive (as in the case  $\beta\rho = 1$ ) but, in addition, the effects on the intergenerational distribution that are implied by the replacement of the utility function have to be taken into account, as well. Since higher saving corresponds to a more equal intergenerational distribution in the case  $\beta\rho < 1$ , the new utility function  $v(c)$  must be more risk averse in order to ensure a higher level of optimal saving (see Proposition 2).<sup>2</sup> In the other case with  $\beta\rho > 1$ , it is the future generation that is privileged by the underlying combination of  $\beta$  and  $\rho$  which is reflected through  $\hat{w}_1 > w_0 - \rho s_u^*$ . To generate higher savings in this situation, the intergenerational distribution has to become less equal such that higher prudence must be combined with less risk aversion (see Proposition 3).

Considering general risk averse utility functions, there is no systematic relationship between changes of prudence and changes of risk aversion,<sup>3</sup> which makes our results substantial. For specific classes of utility functions, however, increased prudence goes along with increased risk aversion such that there are opposing effects. Consider, as an example, the important case of isoelastic utility functions for which the constant elasticity of marginal utility is denoted by  $\eta$ . Further assume that the economy is productive, i.e.  $\rho > 1$ , and that there is no pure time discount such that utility in both periods is given equal weight, i.e.  $\beta = 1$ . An increase in risk aversion  $\eta$  now leads to an increase in the degree of relative prudence which is  $\eta + 1$ . Therefore, the negative impact on savings that then

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<sup>2</sup>For some hints at the importance of risk aversion in this context see Ventura (2007).

<sup>3</sup>See Eeckhoudt and Schlesinger (1994) for examples of the independence and an analysis of some existing relationship between changes of prudence and risk aversion. Additional results on this are in Maggi, Magnani, and Menegatti (2006).

results from higher risk aversion via the consumption smoothing effect over time is counteracted by the precautionary effect that stems from higher prudence. This ambiguity has clearly been noted by Dasgupta (2008) in his comment on Stern (2006).

If future wealth is certain, i.e.  $\tilde{w}_1 = w_1$ , only changes of risk aversion matter. Therefore, by continuity, for any given  $u(c)$ ,  $\beta$ , and  $\rho$  with  $\beta\rho < 1$  and any utility function  $v(c)$  that is more risk averse than  $u(c)$ , there always exists, irrespective of the prudence of  $v(c)$ , a random wealth distribution  $\tilde{w}_1$  with  $E\tilde{w}_1 = w_1$  such that  $s_v^* > s_u^*$ . If  $\beta\rho > 1$ , the analogous result hold for utility functions  $v(c)$  that are less risk averse than  $u(c)$ . In this case, more saving is also compatible with lower prudence if future wealth is uncertain.

Concerning changes of prudence, another irrelevance result is obtained when, for given  $u$ ,  $\beta$ , and  $\rho$ , the condition

$$w_0 - s_u^* \leq \underline{w}_1 + \rho s_u^* \quad (12)$$

holds for  $s_u^*$  and  $\underline{w}_1 := \min \tilde{w}_1$ . Then, with optimal savings, wealth in period 1 in all states of the world is at least as high as wealth in period 0. This clearly requires  $\beta\rho > 1$ , and it is typically possible to generate the situation described in (12) by only decreasing  $\rho$  strongly enough.<sup>4</sup> Now, assume that  $u(c)$  is replaced by any utility function  $v(c)$  that is less risk averse than  $u(c)$ . Then,  $h(c) := v'(c)/u'(c)$  is increasing in  $c$ , such that we get

$$\begin{aligned} v'(w_0 - s_u^*) &= h(w_0 - s_u^*) u'(w_0 - s_u^*) = Eh(w_0 - s_u^*) u'(\tilde{w}_1 + \rho s_u^*) \\ &< Eh(\tilde{w}_1 + \rho s_u^*) u'(\tilde{w}_1 + \rho s_u^*) = Ev'(w_0 + \rho s_u^*). \end{aligned} \quad (13)$$

By the standard argument already applied in the proofs of Propositions 1, 2, and 3 it then follows that  $s_v^* > s_u^*$ , independently of any assumption on the change in prudence. As we have started with a general utility function  $u(c)$ , these considerations also show that the potential irrelevance of changes in prudence for changes in savings is not a remote possibility, but rather a generic phenomenon.

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<sup>4</sup>To see this, let  $u'(c) > 0$  for all  $c > 0$ . Now assume that  $\rho s_u^* < M < \infty$  for all  $\rho > 0$ . Then, from concavity  $Eu'(\tilde{w}_1 + \rho s_u^*) > Eu'(\tilde{w}_1 + M) > 0$  for all  $\rho$  such that, for any  $\beta > 0$ ,  $\lim_{\rho \rightarrow \infty} \beta \rho Eu'(\tilde{w}_1 + \rho s_u^*) = \infty$ . The supposed boundedness of  $\rho s_u^*$ , however, implies  $\lim_{\rho \rightarrow \infty} s_u^* = 0$ , such that  $\lim_{\rho \rightarrow \infty} u'(w_0 - s_u^*) = u'(w_0) < \infty$ , which is not compatible with the first order condition (2). Thus,  $\lim_{\rho \rightarrow \infty} (\underline{w}_1 + \rho s_u^*) = \lim_{\rho \rightarrow \infty} \rho s_u^* = \infty$ . This implies that there must exist a  $\tilde{\rho}$  such that  $\underline{w}_1 + \rho s_u^* > w_0 > w_0 - s_u^*$  for all  $\rho > \tilde{\rho}$ .

## 5 An Impossibility Result

We finally consider the general case where the utility functions in both periods are different. By  $u_0(c_0)$  we denote the utility function in the earlier, and by  $u_1(c_1)$  that in the later period. Under otherwise unchanged assumptions, the objective function then becomes

$$u_0(w_0 - s) + \beta E u_1(\tilde{w}_1 + \rho s). \quad (14)$$

We now show that, given  $u_0(c_0)$ ,  $\beta$ , and  $\rho$ , it is not possible to characterize the class of period 1 utility functions  $v_1(c_1)$  that induces higher savings than the original utility function  $u_1(c_1)$  only by referring to their (absolute) degrees of risk aversion and prudence. This impossibility result follows from the following Proposition.

**Proposition 4.** *Let  $u_1(c_1)$  be replaced by some other utility function  $v_1(c_1)$ . Then, there always exists a utility function  $\tilde{v}_1(c_1)$  which everywhere has the same degree of risk aversion and prudence as  $v_1(c_1)$ , but induces a lower amount of savings than  $u_1(c_1)$ .*

*Proof.* Define  $\tilde{v}_1(c_1)$  as  $\tilde{v}_1(c_1) := \gamma v_1(c_1)$  for some constant  $\gamma > 0$ . Thus,  $\tilde{v}_1(c_1)$  clearly has the same degrees of risk aversion and prudence as  $v_1(c_1)$ . Now, choose  $\gamma$  small enough such that

$$u'_0(w_1 - s_{u_0, u_1}^*) > \beta E \gamma v'_1(\tilde{w}_1 + \rho s_{u_0, u_1}^*) = \beta E \tilde{v}'_1(\tilde{w}_1 + \rho s_{u_0, u_1}^*) \quad (15)$$

where  $s_{u_0, u_1}^*$  denotes optimal savings under the original combination  $(u_0(c_0), u_1(c_1))$  of utility functions. Then, again by the standard argument described in the proof of Proposition 1, savings must decrease when  $u_1(c_1)$  is substituted by  $\tilde{v}_1(c_1)$ .  $\square$

So we see that, because of a level effect, it cannot be expected in the general case that changes of risk aversion and/or prudence will provide sensible results on changes of savings behavior.

## 6 Conclusion

This paper has confirmed that only in rather limited cases changes in the degree of prudence of utility functions have unambiguous effects on optimal saving in the standard two period model. Only when there are identical utility functions in both periods and the underlying combination of the interest rate and the pure discount rate approximately give rise to distributional neutrality across the two

periods, it is ensured that higher prudence induces higher savings. Otherwise, additional properties of the utility functions also play an important role. With identical utility functions in both periods, changes of risk aversion are also relevant when the intergenerational distribution is not balanced. Then, distributional effects that are not grasped by changing prudence but instead by changing risk aversion as a separate determinant become relevant for the saving decision. In general it is, depending on the given interest and pure time discount rate, well possible that the precautionary effect and the consumption smoothing effect over time that result from a change of the utility function either support or work against each other.

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