POPULATION, PENSIONS, AND THE DIRECTION OF TECHNICAL CHANGE

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Abstract: An analytical framework is developed to study the repercussions between endogenous capital- and labor-saving technical change and population aging. Following an intuition often attributed to Hicks (1932), I ask whether and how population aging affects the direction of technical change. I conclude that that population aging increases the relative scarcity of labor with respect to capital. Therefore, there will be more labor- and less capital-augmenting technical change. These evolutions may lead to faster TFP growth during the demographic transition. Institutional characteristics of the PAYGO pension system affect the direction of technical change through their differential effect on savings.

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1 Introduction

Population aging, i. e., the process by which older individuals become a proportionally larger fraction of the total population, is an enduring phenomenon in many of today's developed and developing countries. For instance, Table 1 shows predictions of the United Nations (United Nations (2007)) concerning the old-age dependency ratio for several countries or regions.[1] Roughly speaking, between 2005 and 2050 this ratio is estimated to double in Europe and Northern America. In China, India, and Japan its predicted increase is even more pronounced. To meet this challenge it is necessary to understand the economic consequences of such drastic demographic developments.

The topic of this paper is the role of population aging for economic growth in the presence of a pay-as-you-go pension scheme. The basic idea of my analysis is Hicks' contention according to which innovation incentives depend on the relative scarcity of factors of production (Hicks (1932)). Since the process of population aging tends to reduce the labor force, it may render labor scarcer relative to capital. Accordingly, the relative price of labor increases and labor-saving innovations become more profitable. In Heer and Irmen (2009), we argue that this channel alone leads to faster growth of labor productivity and of total factor productivity. My analysis here extends the framework of Heer and Irmen and allows for labor- *and* capital-augmenting technical change. Since an increase in the relative scarcity of labor comes along with an increase in the relative abundance of capital, the incentives to engage in capital augmenting technical change fall. An important question is then whether the overall effect of population aging on productivity growth and TFP-growth is still positive once the direction of technical change is endogenous.

In a general dynamic equilibrium, the validity of Hicks' contention does not only hinge on the evolution of the labor force. It also depends on the ability and the willingness of an aging population to save and to accumulate capital. The results of Heer and Irmen (2009) suggest that the pension scheme is an influential determinant of the economy's propensity to save. For instance, they find that a pension scheme with a constant contribution rate leads to faster capital accumulation than a scheme with a constant replacement rate since the necessary adjustments of the pension scheme in response to population aging reduce savings be less in the former than in the latter case. Therefore, productivity growth and TFP-growth is fastest under a

[1]These numbers appear in United Nations (2007) as the 'medium variant' prediction The old-age dependency ratio is the ratio of the population aged 65 or over to the population aged 15-64. This ratio is stated as the number of dependants per 100 persons of working age (15-64).

Table 1: Old-Age Dependency Ratios in Selected Countries and Regions.

Year	Europe	Northern America	Germany	United States of America	China	India	Japan
2005	23	18	28	18	11	8	30
2050	48	35	51	34	39	21	74

constant contribution rate. The question is then whether these findings carry over in an environment with capital- and labor-saving technical change.

To address these questions, I devise a new neoclassical growth model with endogenous capital- and labor-augmenting technical change. The model is set up in discrete time. This allows for a representation of population aging in a framework with two-period lived overlapping generations as in Samuelson (1958) or Diamond (1965). My analytical framework is *neoclassical* since it maintains the assumptions of perfect competition, of an aggregate production function with constant returns to scale and positive and diminishing marginal products, and of capital accumulation. It features *endogenous growth* since economic growth results from innovation investments undertaken by profit-maximizing firms. To allow for innovation investments in *capital- and labor-augmenting technical change*, I introduce two intermediate-good sectors, one producing a capital-intensive intermediate, the other a labor-intensive intermediate.[2] Innovation investments increase the productivity of capital and labor at the level of these intermediate-good firms. Moreover, they feed into aggregate productivity indicators that evolve cumulatively, i. e., in a way often referred to as reflecting a 'standing on the shoulders of giants'.

In this framework, I derive the following major results. First, I establish the existence and the local stability properties of a unique steady state with finite stationary variables. Along such a trajectory there is no capital-augmenting technical change.

[2]The production sector extends and complements the one of in Irmen (2005) by allowing for capital-augmenting technical change. In turn, the latter builds on Hellwig and Irmen (2001) and Bester and Petrakis (2003). See Acemoglu (2003b) for an alternative model of endogenous capital- and labor-augmenting technical change where innovation investments are financed through rents that accrue in an environment with monopolistic competition and the household side is modeled following Ramsey (1928). Acemoglu's and my model build on ideas developed by the so-called 'induced innovations' literature of the 1960s which is surveyed in Acemoglu (2003a). Also, see Funk (2002) for a critique of this literature and a microfoundation in a perfectly competitive setup.

Per-capita magnitudes such as consumption, savings, or the real wage grow at the growth rate of labor-augmenting technical change. These findings are in line with the so-called Steady-State Growth Theorem of Uzawa (1961).

Second, I study the economic consequences of population aging. In my framework, this process may be captured by a once and for all decrease in the population growth rate or by a sequence of ever declining population growth rates. In both cases, the economy leaves its steady state and embarks on a trajectory with an increased speed of capital deepening, i. e., labor becomes scarcer relative to capital. In line with Hicks' contention, there is more labor- and less capital-augmenting technical change. As a result, the growth rate of labor productivity increases whereas the one of capital falls.[3] Moreover, these adjustments in the direction of technical change may increase the growth rate of the total factor productivity (TFP). At later stages of the transition, the growth rate of the economy converges to its initial level. Thus, in the long run, a permanent decline in population growth does not affect the economy's growth rate. However, the initial phase with capital deepening at an increasing speed may be longer, if I allow for the population growth rate to decline continuously along the transition.

These findings suggest that in the presence of capital-augmenting technical change the long-run predictions that appear in Heer and Irmen (2009) do not hold true. However, along the transition of, say 50 to 100 years, their results are likely to be unaffected.

Third, I turn to the effect of the pension scheme on the direction of technical change. I consider a pay-as-you-go pension scheme with a constant replacement rate. I establish the existence of a unique steady state and show that the steady-state growth rate of the economy is not affected by the presence or the introduction of such a scheme. However, if such a scheme is introduced the economy leaves its steady state and embarks on a path with a reduced speed of capital deepening. Hence, initially there is more capital- and less labor saving technical change. Intuitively, the pension scheme reduces the incentives to save. Therefore, there will be less capital accumulation, capital becomes relatively scarcer, and the direction of technical change shifts towards more capital-augmenting technical change.

Fourth, I analyze the role of the pension scheme for the interaction between population aging and the direction of technical change. I find that the qualitative properties

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[3]Cutler, Poterba, Sheiner, and Summers (1990) provide evidence of a negative and significant partial correlation between labor productivity growth and labor force growth in cross-country regressions. Barro and Sala-í-Martin (2004), Kelley and Schmidt (1995), and Kormendi and Meguire (1985) run cross-country growth regressions and find a negative and significant partial correlation between population growth and growth of per-capita GDP.

of the transition ignited by a once and for all decline in the population growth rate are the same with and without a pension scheme. However, my comparative institutional analysis reveals that the initial effect on faster capital deepening is more pronounced if the pension scheme has a constant contribution rate as compared to a constant replacement rate.

My research builds on and contributes to a recent and growing literature that aims at an understanding of the economic consequences of population aging in different settings. To the best of my knowledge, the implications of the demographic transition on the direction of technical change and the role of the pension scheme for these implications have not yet received proper attention. The purpose of this paper is to close this gap.

Existing studies that analyze population aging from an endogenous growth perspective include Ludwig, Schelke, and Vogel (2008) and Heer and Irmen (2009). The former considers endogenous human capital investments of the households and a growth mechanism along the lines of Lucas (1988). The latter focusses on labor-augmenting technical change alone, i. e., the direction of technical change is exogenous. In spite of their theoretical contributions, both of these papers contribute to a large and growing literature that analyzes population aging in large-scale heterogeneous overlapping generations models in the spirit of Auerbach and Kotlikoff (1987). Important other contributions of this branch that do not consider aspects of endogenous growth include İmrohoroğlu, İmrohoroğlu, and Joines (1995) and de Nardi, İmrohoroğlu, and Sargent (1999).

This paper is organized as follows. Section 2 presents the details of the model. Section 3 studies the intertemporal general equilibrium and establishes the existence and the stability properties of the steady state. In Section 4, I study the implications of population aging on economic growth. The focus of Section 5 is on the role of the pension scheme for the direction of technical change. Section 6 draws the threads of the previous two sections together as studies the interaction between population aging, pensions, and growth. Section 7 concludes. Proofs are relegated to Appendix A. Details on the phase diagrams are given in Appendix B. Appendix C presents the results of three numerical calibration exercises.

2 The Basic Model

The economy has a household sector, a final-good sector, and an intermediate-good sector in an infinite sequence of periods $t = 1, 2, ..., \infty$. The household sector comprises two-period lived individuals as in Samuelson (1958) or Diamond (1965).

There are six objects of exchange. The *manufactured final good* can be consumed or invested. If invested it may either become future capital or serve as an input in current innovation activity undertaken by intermediate-good firms. Intermediate-good firms produce one of two types of intermediates and sell it to the final-good sector. The production of the *labor-intensive intermediate good* uses labor as the sole input, the only input in the production of the *capital-intensive intermediate good* is capital. Labor- and capital-saving technical change is the result of innovation investments undertaken by the respective intermediate-good firms. The household sector supplies *labor* and *capital* to the intermediate-good sector. Labor is "owned" by the young, capital is owned by the old. Finally, there are *bonds* with a maturity of one period. Each period has markets for all six objects of exchange. The final good serves as numéraire.

2.1 Households

The household sector consists of two-period lived individuals. They work and save when young, retire when old, and consume during both periods of their life. At t, there are L_t young and L_{t-1} old individuals. Individual labor supply when young is exogenous and normalized to one. Unless indicated otherwise, the growth rate of the labor force, $\lambda > (-1)$, is constant over time. Then, it coincides with the population growth rate.

Lifetime utility of a member of cohort t is

$$U_t = u\left(c_t^y\right) + \beta u\left(c_{t+1}^o\right), \tag{2.1}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a per-period utility function. It is $\mathcal{C}^2$ and satisfies $u'(c) > 0 > u''(c)$ as well as $\lim_{c \to 0} u'(c) = \infty$. Consumption of a member of cohort t when young and old is c_t^y and c_{t+1}^o, respectively. Moreover, $\beta \in (0,1)$ is the discount factor.

The maximization of (2.1) is subject to the per-period budget constraints $c_t^y + s_t = w_t$ and $c_{t+1}^o = s_t(1 + r_{t+1})$, where s_t denotes savings, $w_t > 0$ the real wage at t, and $r_{t+1} > (-1)$ the (expected) real interest rate on bonds coming due at $t+1$. The optimal plan of a member of cohort t, $\left(c_t^y, s_t, c_{t+1}^o\right)$, results from the Euler condition

$$-u'\left(c_t^y\right) + \beta\left(1 + r_{t+1}\right)u'\left(c_{t+1}^o\right) = 0 \tag{2.2}$$

in conjunction with the two budget constraints. Given our assumptions on preferences, this plan involves a continuous and partially differentiable function that relates savings to current income and the real interest rate

$$s_t = s\left[w_t, r_{t+1}\right]. \tag{2.3}$$

To derive analytical results I shall mostly assume log-utility such that $U_t = \ln c_t^y + \beta \ln c_{t+1}^o$. Then, savings are

$$s_t = \frac{\beta}{1+\beta} w_t \tag{2.4}$$

and do no longer depend on the expected real interest rate due to compensating income and substitution effects. More importantly, this assumption offers the advantage that the dynamical system of the economy under scrutiny has only two instead of three state variables (see the proof of Proposition 2 for details). Therefore, the analysis of the comparative dynamics may rely on two-dimensional phase diagrams.

2.2 The Final-Good Sector

The final-good sector produces with a neoclassical production function $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$

$$Y_t = F\left(Y_{K,t}, Y_{L,t}\right), \tag{2.5}$$

where Y_t is aggregate output in t, $Y_{K,t}$ is the aggregate amount of the capital-intensive intermediate input used in t, and $Y_{L,t}$ denotes the aggregate amount of the labor-intensive intermediate input. The function F is $\mathcal{C}^2$ with $F_1 > 0 > F_{11}$ and $F_2 > 0 > F_{22}$. Moreover, it exhibits constant-returns-to-scale. with respect to both inputs.

Under these assumptions, we can study the competitive behavior of the final-good sector in terms of a single representative firm. In units of the final good of period t as numéraire the profit in t of the final-good sector is

$$Y_t - p_{K,t} Y_{K,t} - p_{L,t} Y_{L,t}, \tag{2.6}$$

where $p_{j,t}$, $j = K$ or L is the price of the respective intermediate.

The final-good sector takes the sequence $\{p_{K,t}, p_{L,t}\}_{t=1}^{\infty}$ of factor prices as given and maximizes the sum of the present discounted values of profits in all periods. Since it simply buys both intermediates in each period, its maximization problem is equivalent to a series of one-period maximization problems. Define the period-t factor intensity in the final-good sector as

$$\kappa_t \equiv \frac{Y_{K,t}}{Y_{L,t}}. \tag{2.7}$$

Then, the production function in intensive form is $F(\kappa_t, 1) \equiv f(\kappa_t)$. The respective profit-maximizing first-order conditions for $t = 1, 2, ...$ are

$$Y_{K,t} \quad : \quad p_{K,t} = f'(\kappa_t), \tag{2.8}$$

$$Y_{L,t} \quad : \quad p_{L,t} = f(\kappa_t) - \kappa_t f'(\kappa_t). \tag{2.9}$$

2.3 The Intermediate-Good Sector

There are two different sets of intermediate-good firms, each represented by the set $\mathbb{R}_+$ of nonnegative real numbers with Lebesgue measure. Intermediate-good firms may either belong to the sector that produces the labor- or the capital-intensive intermediate. In other words, they are part of the labor- or the capital-intensive intermediate-good sector.

2.3.1 Technology

At any date t, all firms of a sector have access to the same sector-specific technology with production function

$$y_{l,t} = \min\{1, a_t l_t\} \quad \text{and} \quad y_{k,t} = \min\{1, b_t k_t\}, \tag{2.10}$$

where $y_{l,t}$ and $y_{k,t}$ is output, 1 a capacity limit,[4] a_t and b_t denote the firms' labor and capital productivity in period t, l_t and k_t is the labor and the capital input. The firms' respective labor and capital productivity is equal to

$$a_t = A_{t-1}(1 - \delta + q_t^A) \quad \text{and} \quad b_t = B_{t-1}(1 - \delta + q_t^B); \tag{2.11}$$

here A_{t-1} and B_{t-1} denote aggregate indicators of the level of technological knowledge to which innovating firms in period t have access for free. Naturally, $\delta \in (0, 1)$ is the rate of depreciation of technological knowledge in both sectors, and q_t^A and q_t^B are indicators of (gross) productivity growth at the firm.

To achieve a productivity growth rate $q_t^j > 0$, $j = A, B$, a firm must invest $i(q_t^j)$ units of the final good in period t. The function $i : \mathbb{R}_+ \to \mathbb{R}_+$ is the same for

[4]The analysis is easily generalized to allow for an endogenous capacity choice requiring prior capacity investments, with investment outlays a strictly convex function of capacity. In such a setting profit-maximizing behavior implies that a larger innovation investment is accompanied by a larger capacity investment (see, Hellwig and Irmen (2001) for details). Thus, the simpler specification treated here abstracts from effects on firm size in an environment with changing levels of innovation investments. Otherwise, it does not affect the generality of our results.

both sectors, time invariant, $\mathcal{C}^2$, and strictly convex. Moreover, with the notation $i'(q^j) \equiv di(q^j)/dq^j$ for $j = A, B$, it satisfies

$$i(0) = i'(0) = 0, \quad i'\left(q^j\right) > 0, \quad \lim_{q^j \to \infty} i\left(q^j\right) = \lim_{q^j \to \infty} i'\left(q^j\right) = \infty. \tag{2.12}$$

Hence, higher rates of productivity growth require ever larger investments.

If a firm innovates the assumption is that an innovation in period t is proprietary knowledge of the firm only in t, i. e., in the period when the innovation materializes. Subsequently, the innovation becomes embodied in the sector specific productivity indicators (A_t, B_t), (A_{t+1}, B_{t+1}), ..., with no further scope for proprietary exploitation. The evolution of these indicators will be specified below. If firms decide not to undertake an innovation investment in period t, then they have access to the production technique represented by A_{t-1} and B_{t-1} such that $a_t = A_{t-1}(1-\delta)$ and $b_t = B_{t-1}(1-\delta)$, respectively.

2.3.2 Profit Maximization and Zero-Profits

Per-period profits in units of the current final good are

$$\pi_{L,t} = p_{L,t}y_{l,t} - w_t l_t - i(q_t^A), \quad \pi_{K,t} = p_{K,t}y_{k,t} - R_t k_t - i(q_t^B), \tag{2.13}$$

where $p_{L,t}y_{l,t}$, $p_{K,t}y_{k,t}$ is the respective firm's revenue from output sales, $w_t l_t$, $R_t k_t$ its wage bill at the real wage rate w_t and its capital cost at the real rental rate of capital R_t, and $i(q_t^j)$, $j = A, B$, its outlays for innovation investment.

Firms choose a production plan $(y_{l,t}, l_t, q_t^A)$ and $(y_{k,t}, k_t, q_t^B)$ taking the sequence $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$ of real prices and the sequence $\{A_{t-1}, B_{t-1}\}_{t=1}^{\infty}$ of aggregate productivity indicators as given. They choose a production plan that maximizes the sum of the present discounted values of profits in all periods. Because production choices for different periods are independent of each other, for each period t, they choose the plan $(y_{l,t}, l_t, q_t^A)$ and $(y_{k,t}, k_t, q_t^B)$ that maximizes the profit $\pi_{L,t}$ and $\pi_{K,t}$, respectively.

If a firm innovates, it incurs an investment cost $i(q_t^j) > 0$ that is associated with a given innovation rate $q_t^j > 0$ and is independent of the output $y_{l,t}$ or $y_{k,t}$. An innovation investment is only profit-maximizing if the firm's margin is strictly positive, i. e., if $p_{L,t} > w_t/a_t$ or $p_{K,t} > R_t/b_t$. Then, there is a positive scale effect, namely if the firm innovates, it wants to apply the innovation to as large an output as possible and produces at the capacity limit, i. e., $y_{l,t} = 1$ or $y_{k,t} = 1$. The choice of (l_t, q_t^A)

and (k_t, q_t^B) must then minimize the costs of producing the capacity output, i. e., assuming $w_t > 0$ and $R_t > 0$ these input combinations must satisfy

$$l_t = \frac{1}{A_{t-1}(1-\delta+q_t^A)}, \quad k_t = \frac{1}{B_{t-1}(1-\delta+q_t^B)}, \tag{2.14}$$

and

$$\begin{aligned} \hat{q}_t^A &\in \arg\min_{q^A \geq 0} \left[\frac{w_t}{A_{t-1}(1-\delta+q^A)} + i(q^A) \right], \\ \hat{q}_t^B &\in \arg\min_{q^B \geq 0} \left[\frac{R_t}{B_{t-1}(1-\delta+q^B)} + i(q^B) \right]. \end{aligned} \tag{2.15}$$

Given the convexity of the innovation cost function and the fact that $i'(0) = 0$, the following conditions determine a unique level $\hat{q}_t^A > 0$ and $\hat{q}_t^B > 0$ as the solution to the first-order conditions

$$\frac{w_t}{A_{t-1}(1-\delta+\hat{q}_t^A)^2} = i'\left(\hat{q}_t^A\right) \quad \text{and} \quad \frac{R_t}{B_{t-1}(1-\delta+\hat{q}_t^B)^2} = i'\left(\hat{q}_t^B\right). \tag{2.16}$$

The latter relate the marginal reduction of a firm's wage bill/capital cost to the marginal increase in its investment costs.

If a firm's margin is not strictly positive, i. e., $p_{L,t} \leq w_t/a_t$ or $p_{K,t} \leq R_t/b_t$, then it will not invest. In case of a zero-margin, any plan $(y_{l,t}, l_t, 0)$ with $y_{l,t} \in [0,1]$ and $l_t = y_{l,t}/A_{t-1}(1-\delta)$ or $(y_{k,t}, l_t, 0)$ with $y_{k,t} \in [0,1]$ and $k_t = y_{k,t}/B_{t-1}(1-\delta)$ maximizes π_t^L or π_t^K, respectively. Without loss of generality and to simplify the notation, we shall assume for this case that entering firms still produce the capacity output. If a firm faces a strictly negative margin, it won't enter and $(0,0,0)$ is the optimal plan.

2.4 Consolidating the Production Sector

Turning to the implications for the general equilibrium, recall that the set of each sector of intermediate-good firms is $\mathbb{R}_+$ with Lebesgue measure. Consider equilibria where both intermediates are produced. Then, the maximum profit of any intermediate-good firm producing the labor- or the capital-intensive intermediate must be zero at any t. Indeed, since the supply of labor and capital is bounded in each period, the set of intermediate-good firms employing more than some $\varepsilon > 0$ units of labor or capital must have bounded measure and hence must be smaller than the set of all intermediate-good firms. Given that inactive intermediate-good firms must be maximizing profits just like the active ones, I need that maximum profits of all active intermediate-good firms at equilibrium prices are equal to zero.

Using (2.13), (2.14), and (2.16), we find for profit-maximizing intermediate-good firms earning zero profits in equilibrium that

$$p_{L,t} = (1-\delta+\hat{q}_t^A)i'(\hat{q}_t^A) + i(\hat{q}_t^A), \quad p_{K,t} = (1-\delta+\hat{q}_t^B)i'(\hat{q}_t^B) + i(\hat{q}_t^B), \tag{2.17}$$

i. e., the price is equal to variable cost plus fixed costs when w_t/a_t and R_t/b_t are consistent with profit-maximization as required by (2.16). Upon combining the equilibrium conditions of the final-good sector and both intermediate-good sectors we find the following proposition.

Proposition 1 *If (2.8), (2.9), and (2.17) hold as equalities for all producing firms at t, then there are maps, $g^A : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $g^B : \mathbb{R}_{++} \to \mathbb{R}_{++}$, such that $\hat{q}_t^A = g^A(\kappa_t)$ and $\hat{q}_t^B = g^B(\kappa_t)$ satisfy*

$$g_\kappa^A(\kappa_t) > 0 \quad \textit{and} \quad g_\kappa^B(\kappa_t) < 0 \qquad \textit{for all } \kappa_t > 0. \tag{2.18}$$

Proposition 1 states a key property of the production sector. The equilibrium incentives to engage in labor- and capital-saving technical change depend on the factor intensity of the final-good sector. This is due to the properties of the production function F. Under constant returns to scale and positive, yet decreasing marginal products, both inputs are complements. Hence, having relatively more of $Y_{K,t}$, i. e., a higher κ_t, increases $p_{L,t}$ and decreases $p_{K,t}$. These price movements increase $\hat{q}_t^A$ and decrease $\hat{q}_t^B$ in accordance with (2.17).

2.5 Evolution of Technological Knowledge

The evolution of the economy's level of technological knowledge is given by the evolution of the aggregate indicators A_t and B_t. An important question is then how these indicators are linked to the innovation investments of individual intermediate-good firms.

Labor- and capital-saving productivity growth occurs at those intermediate-good firms that produce at t. Denoting the measure of the firms producing the labor- and the capital-intensive intermediate by n_t and m_t, respectively, their contribution to A_t and B_t is equal to the highest level of labor and capital productivity attained by one of them, i. e.,

$$\begin{aligned} A_t &= \max\{a_t(n) = A_{t-1}\left(1-\delta+q_t^A(n)\right) \mid n \in [0, n_t]\} \\ B_t &= \max\{b_t(m) = B_{t-1}\left(1-\delta+q_t^B(m)\right) \mid m \in [0, m_t]\}. \end{aligned} \tag{2.19}$$

Since in equilibrium $q_t^A(n) = q_t^A$ and $q_t^B(m) = q_t^B$, we have $a_t = A_{t-1}\left(1 - \delta + q_t^A\right)$ and $b_t = B_{t-1}\left(1 - \delta + q_t^B\right)$. Hence, for all $t = 1, 2, ...$

$$A_t = A_{t-1}\left(1 - \delta + q_t^A\right) \quad \text{and} \quad B_t = B_{t-1}\left(1 - \delta + q_t^B\right) \tag{2.20}$$

with $A_0 > 0$ and $B_0 > 0$ as initial conditions.

3 Intertemporal General Equilibrium

3.1 Definition

A *price system* corresponds to a sequence $\{p_{L,t}, p_{K,t}, r_t, w_t, R_t\}_{t=1}^{\infty}$. An *allocation* is a sequence $\{c_t^y, s_t, c_t^o, Y_t, Y_{K,t}, Y_{L,t}, n_t, m_t, y_{L,t}, y_{K,t}, q_t^A, q_t^B, a_t, b_t, l_t, k_t, L_{t-1}, K_t\}_{t=1}^{\infty}$ that comprises a strategy $\{c_t^y, s_t, c_{t+1}^o\}_{t=1}^{\infty}$ for all cohorts, consumption of the old at $t = 1$, c_1^o, a strategy $\{Y_t, Y_{K,t}, Y_{L,t}\}_{t=1}^{\infty}$ for the final-good sector, measures n_t and m_t of intermediate-good firms active at t producing the capacity output $y_{L,t} = y_{K,t} = 1$ with input choices (l_t, q_t^A), and (k_t, q_t^B) resulting in the respective productivity levels (a_t, b_t), and demanding the aggregate supply of labor and capital, L_t and K_t.

For an exogenous evolution of the labor force, $L_t = L_0 (1 + \lambda)^t$ with $L_0 > 0$ and $\lambda > (-1)$, a given initial level of capital, $K_1 > 0$, and initial values of technological knowledge, $A_0 > 0$ and $B_0 > 0$, an *equilibrium* corresponds to a price system, an allocation, and a sequence $\{A_{t-1}, B_{t-1}\}_{t=1}^{\infty}$ of indicators for the level of aggregate technological knowledge that satisfy the following conditions for all $t = 1, 2, ..., \infty$:

(E1) The young of each period save according to (2.4) and supply L_t units of labor.

(E2) The production sector satisfies Proposition 1.

(E3) The market for the final good clears, i. e.,

$$L_{t-1}c_t^o + L_t c_t^y + I_t^K + I_t^A + I_t^B = Y_t, \tag{3.1}$$

where I_t^K is capital investment, I_t^A and I_t^B denote aggregate innovation investments in labor- and capital-augmenting technical change.

(E4) The market for both intermediates clears, i. e.,

$$Y_{L,t} = n_t \quad \text{and} \quad Y_{K,t} = m_t. \tag{3.2}$$

(E5) There is full employment of labor and capital such that

$$n_t l_t = L_t \quad \text{and} \quad m_t k_t = K_t. \tag{3.3}$$

(E6) Bonds are in zero net supply, and

$$1 + r_t = R_t. \tag{3.4}$$

(E7) The productivity indicators A_t and B_t evolve according to (2.20).

(E1) and (E2) guarantee optimal behavior of all households and firms at t. Since the old at $t = 1$ own the capital stock, their consumption is $L_0 c_1^o = R_1 K_1$. Due to constant returns to scale in final-good production, there are no profits in equilibrium. The resource constraint (3.1) requires savings to equal capital investment, i. e.,

$$s_t L_t = K_{t+1} \quad \text{for } t = 1, 2, ..., \infty. \tag{3.5}$$

Market clearing for each intermediate good (E4), full employment of labor and capital (E5), (2.14), and the updating condition (E7) imply that $n_t = A_t L_t$ and $m_t = B_t K_t$. Hence, in equilibrium, we have

$$Y_{L,t} = A_t L_t \quad \text{and} \quad Y_{K,t} = B_t K_t, \tag{3.6}$$

$$I_t^A = A_t L_t i\left(q_t^A\right) \quad \text{and} \quad I_t^B = B_t K_t i\left(q_t^B\right), \tag{3.7}$$

i. e., aggregate output of each intermediate-good is equal to the respective input in efficiency units, and aggregate investment in labor- and capital-augmenting technical change is proportionate to the respective input in efficiency units. Observe that (3.6) and (2.7) imply an equilibrium factor intensity

$$\kappa_t = \frac{B_{t-1}\left(1 - \delta + g^B\left(\kappa_t\right)\right) K_t}{A_{t-1}\left(1 - \delta + g^A\left(\kappa_t\right)\right) L_t} \tag{3.8}$$

$$= \frac{B_t K_t}{A_t L_t}. \tag{3.9}$$

Thus, in equilibrium κ_t is the intensity of efficient capital per unit of efficient labor. For short, I shall henceforth refer to it as the 'efficient capital intensity' as opposed to the 'capital intensity', i. e., the amount of capital per worker K_t/L_t. This finding allows for a reinterpretation of Proposition 1 from a perspective of the general equilibrium.

Corollary 1 *At all t, the equilibrium incentives to engage in capital- and labor-saving technical change depend on K_t/L_t, i. e., on the relative scarcity of labor with respect to capital. If K_t/L_t increases, then $\hat{q}_t^A = g^A\left(\kappa_t\right)$ increases and $\hat{q}_t^B = g^B\left(\kappa_t\right)$ decreases and vice versa.*

Hence, if labor is scarcer relative to capital then there is less capital- and more labor-saving technical change. Corollary 1 confirms this famous intuition often attributed to Hicks (1932). The formal argument comes from (3.8) and Proposition 1. For higher values of K_t/L_t the implied κ_t increases since $g_\kappa^A(\kappa_t) > 0 > g_\kappa^B(\kappa_t)$. Accordingly, $\hat{q}_t^A$ increases and $\hat{q}_t^B$ decreases.

Finally, note that the interest rate is pegged to the rental rate of capital according to (3.4) since bonds and capital are perfect substitutes and capital fully depreciates after one period.

3.2 The Dynamical System

The following proposition shows that the evolution of the economy can be characterized by means of two state variables, namely the factor intensity of the final-good sector, κ_t, and the level of capital-augmenting technological knowledge, B_t.

For further reference it proves useful to introduce some notation for stationary variables. I denote the real wage per efficiency unit and the real rental rate of capital per efficiency unit by $w_t/A_t \equiv \tilde{w}(\kappa_t)$ and $R_t/B_t \equiv \tilde{R}(\kappa_t)$. Moreover, I define

$$G(\kappa_t) \equiv \frac{1-\delta+g^A(\kappa_t)}{1-\delta+g^B(\kappa_t)}\kappa_t. \tag{3.10}$$

Proposition 2 *(Dynamical System)*

Given $(K_1, L_1, A_0, B_0) > 0$ *as initial conditions, there is a unique sequence* $\{\kappa_t, B_t\}_{t=1}^{\infty}$ *determined by*

$$\frac{\beta}{(1+\beta)(1+\lambda)}\tilde{w}(\kappa_t) = \frac{G(\kappa_{t+1})}{B_t}, \tag{3.11}$$

and

$$B_{t+1} = B_t\left(1-\delta+g^B(\kappa_{t+1})\right), \tag{3.12}$$

where κ_1 *and* B_1 *satisfy*

$$\kappa_1 = \frac{B_0\left(1-\delta+g^B(\kappa_1)\right)K_1}{A_0\left(1-\delta+g^A(\kappa_1)\right)L_1} > 0, \quad B_1 = B_0\left(1-\delta+g^B(\kappa_1)\right) > 0. \tag{3.13}$$

According to Proposition 2, the dynamical system is a two-dimensional system of first-order, autonomous, non-linear difference equations. It involves an equation of motion for the capital intensity (3.11) and for the level of capital-augmenting technological knowledge (3.12). For any given pair $(\kappa_t, B_t) \in \mathbb{R}_{++}$, (3.11) assigns a unique value κ_{t+1} which is then used to find B_{t+1} with (3.12). Since K_1 is an initial condition, κ_1 adjusts in a way consistent with equilibrium behavior and its definition.

Define a steady state as a trajectory along which all magnitudes grow at a constant rate. From (3.12) I deduce that a trajectory with $B_{t+1}/B_t - 1 = const.$ requires $\kappa_t = \kappa_{t+1} = \kappa^*$. Moreover, according to (3.11), the latter needs $B_{t+1} = B_t = B^*$. Hence, a steady state is a solution to

$$\frac{\beta}{(1+\beta)(1+\lambda)} \tilde{w}(\kappa^*) = \frac{G(\kappa^*)}{B^*}. \tag{3.14}$$

$$g^B(\kappa^*) = \delta \tag{3.15}$$

Proposition 3 *(Steady State)*

1. *There is a unique steady state involving $\kappa^* \in (0, \infty)$ and $B^* \in (0, \infty)$ if and only if*

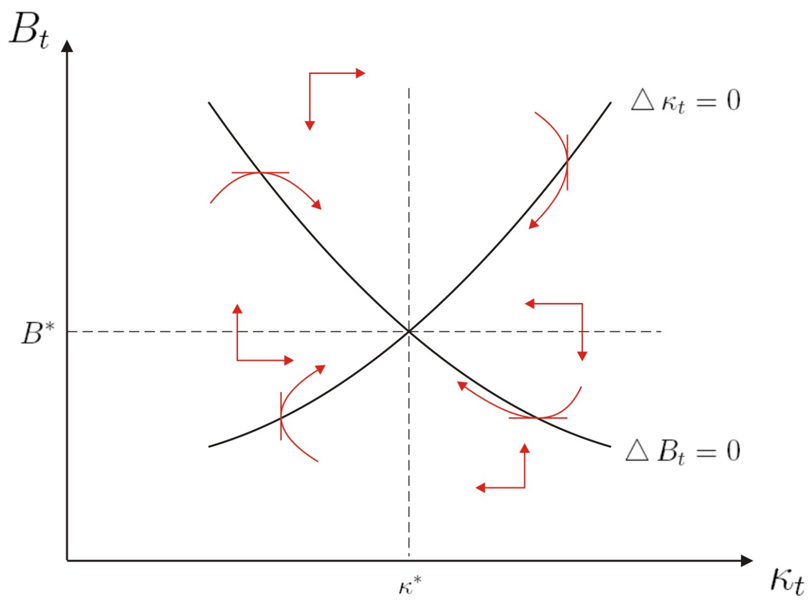
$$\lim_{k \to 0} g^B(\kappa) > \delta > \lim_{k \to \infty} g^B(\kappa). \tag{3.16}$$

2. *Along the steady state, the real interest rate is constant at $r^* = B^*[f'(\kappa^*) - i(\delta)] - 1$, the real wage, w_t, and per-capita magnitudes such as c_t^y, c_t^o, s_t grow at rate $g^* \equiv A_{t+1}/A_t - 1 = g^A(\kappa^*) - \delta$.*

According to Proposition 3, a steady state exists if the incentives to engage in capital-augmenting technical change are sufficiently high as $\kappa \to 0$, and sufficiently low as $\kappa \to \infty$. In light of Proposition 1, these conditions are linked to the marginal product of the capital-intensive intermediate good in final-good production. If this product is sufficiently high (low) when $Y_K \to 0$ ($Y_K \to \infty$), then a steady state exists. Condition (3.16) would always hold if I had imposed the usual Inada conditions on the final-good production function F.

Contrary to the one-sector neoclassical growth model where multiple steady states may arise if the elasticity of substitution is sufficiently small (see, e. g., Galor (1996)), the steady state is unique in my setting. The reason is that here the steady-state capital intensity κ^* is not determined by the difference equation for capital accumulation. Instead, κ^* adjusts such that additions to the stock of capital-augmenting technological knowledge just offset depreciation. Diminishing returns to capital still play a key role here since they imply $g_\kappa^B(\kappa_t) < 0$. Given κ^*, the steady-state level of capital-augmenting technological knowledge B^* adjusts to satisfy (3.14).

Figure 3.1: The Phase-Diagram of a Stable Node (κ^*, B^*).



The steady-state growth rate of the economy is equal to the growth rate of the stock of labor-augmenting technological knowledge.[5] All per-capita magnitudes grow at this rate. There is no growth of capital-augmenting technical change. These findings are consistent with the so-called Steady-State Growth Theorem of Uzawa (1961).[6] Due to the presence of depreciation of technological knowledge, the growth rate of the economy, g^*, need not be positive at κ^*.[7]

To get an idea of the transitional dynamics involved, consider the phase diagram in the (B_t, κ_t) – plane shown in Figure 3.1.[8] Based on (3.11), I denote the locus of all pairs of (B_t, κ_t) for which the evolution of κ is at a point of rest by $\Delta\kappa_t =$

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[5]To be complete, I note that along the steady state, we have $Y_{t+1}/Y_t = Y_{K,t+1}/Y_{K,t} = Y_{L,t+1}/Y_{L,t} = K_{t+1}/K_t = n_{t+1}/n_t = m_{t+1}/m_t = \left(1-\delta+g^A(\kappa^*)\right)(1+\lambda)$. Moreover, $p_L^* = f(\kappa^*) - \kappa^* f'(\kappa^*)$, $p_K^* = f'(\kappa^*)$, $R^* = B^*\left[p_K^* - i(\delta)\right]$. Finally, $k^* = 1/B^*$ and $l_{t+1}/l_t = 1/\left(1-\delta+g^A(\kappa^*)\right)$.

[6]It can be shown that the production side of the economy under scrutiny here is isomorphic to the environment to which Uzawa's theorem applies (see, Irmen (2009)) for the proof of this claim). See Schlicht (2006) and Jones and Scrimgeour (2008) for a recent and very elegant proof of Uzawa's theorem.

[7]To guarantee $g^* > 0$ one needs to impose more structure. For instance, if $\lim_{\kappa\to\infty} g^A(\kappa) = \infty$, which is the case if the final-good production function F satisfies the Inada conditions, a sufficiently small level of δ assures positive growth such that $g^A(\kappa^*) > g^B(\kappa^*) = \delta$.

[8]Details on the underlying computations can be found in Appendix B.

$0 \equiv \{(B_t, \kappa_t) \,|\kappa_{t+1} - \kappa_t = 0\}$. I assume this locus to be stable in a sufficiently small neighborhood of the steady state. As a consequence, the horizontal arrows point towards $\Delta\kappa_t = 0$. Moreover, this locus is strictly increasing. This assumption can be justified with reference to the one-sector neoclassical growth model where the focus would be confined to locally stable steady states. The locus $\Delta B_t = 0 \equiv \{(B_t, \kappa_t) \,|B_{t+1} - B_t = 0\}$ gives all pairs (B_t, κ_t), for which the evolution of B is at a point of rest. In the vicinity of the steady state, this locus is strictly decreasing and stable. Hence, the vertical arrows point towards $\Delta B_t = 0$. As to the local stability of the steady state the following lemma holds.

Lemma 1 *If the locus $\Delta\kappa_t = 0$ is stable in a vicinity of $(\kappa^*, B^*,)$, then the steady state is either a stable node or a clockwise spiral sink.*

Appendix C presents the result of three calibration exercises where F has a constant elasticity of substitution and i is quadratic. For reasonable parameter values, the model delivers steady-state growth rates that are broadly consistent with the long-run growth performance of industrialized countries. In all three cases the steady state is a stable node.

4 Population and Economic Growth

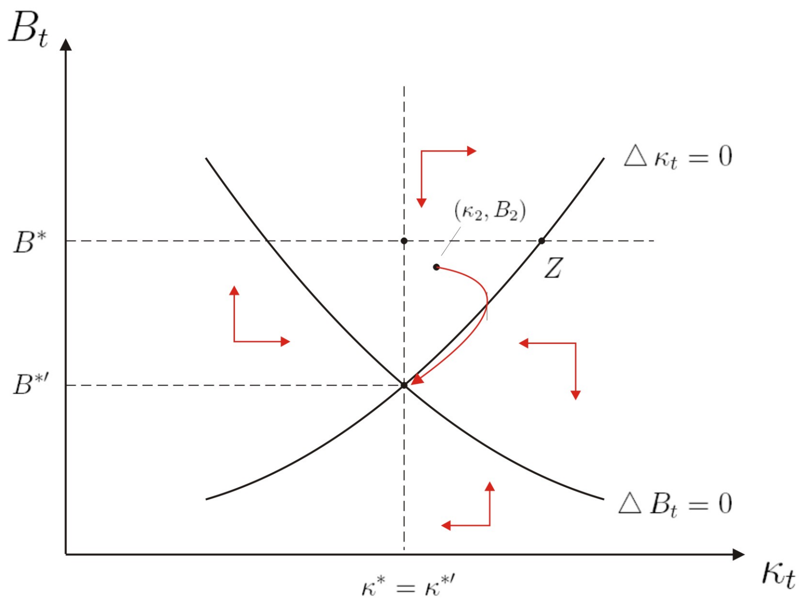
This section studies the role of a changing population growth rate on the steady state and the transitional dynamics in its neighborhood. The key findings appear in the following proposition.

Proposition 4 *(Comparative Statics and Dynamics)*

Consider an economy in a steady state in period $t = 1$. Then, it experiences a permanent decline in its labor force growth rate such that $L_t = L_1(1 + \lambda')^{t-1}$ with $\lambda > \lambda' > (-1)$ for all $t = 2, 3, ..., \infty$. Denote $(\kappa^{*\prime}, B^{*\prime})$ the steady state associated with λ'.

1. *It holds that $\kappa^{*\prime} = \kappa^*$ and $B^{*\prime} < B^*$.*
2. *It holds that $\kappa_2 > \kappa^*$ and $B_2 < B^*$. Suppose λ' is such that (κ_2, B_2) is in a sufficiently small neighborhood of $(\kappa^{*\prime}, B^{*\prime})$. Then, at $t = 2$ the economy embarks on a trajectory $\{\kappa_t, B_t\}_{t=2}^{\infty}$ with $\lim_{t\to\infty} \{\kappa_t, B_t\} = (\kappa^{*\prime}, B^{*\prime})$.*

Figure 4.1: Comparative Statics and Dynamics for a Permanent Decline in the Growth Rate of the Labor Force. The Case of a Stable Node.



Hence, a permanent decline in the growth rate of the labor force does not affect the steady-state capital intensity. This is evident from (3.15) since g^B does not dependent on population growth. Therefore, the steady-state growth rate is also unaffected. However, there is a level effect on capital-augmenting technological knowledge. From (3.14), I deduce that B^* declines at the same rate as the population growth factor.

To understand this result consider the phase diagram of Figure 4.1. At B^*, the permanent decline in population growth shifts the $\Delta\kappa_t = 0-$locus to the right. In a world without capital-augmenting technical change, the new steady state would be at point Z since the evolution of κ_t is stable around (κ^*, B^*). Intuitively, this shift is due to two effects. First, there is what Cutler, Poterba, Sheiner, and Summers (1990) call the *Solow effect*: a smaller population growth rate induces capital deepening, i. e., K_t/L_t increases. Second, there is a *Hicks effect* on labor-augmenting technical change. Even if $B_t = B^*$ is a constant, capital deepening implies a greater κ_t according to Corollary 1. Hence, the incentives to engage in labor-augmenting technical change increase. This is the logic behind the steady-state analysis that appears in Heer and Irmen (2009). As a consequence, their framework predicts that a permanent decline in the population growth rate induces faster steady-state growth due to the Hicks effect on labor-augmenting technical change. However, once I endogenize the direction of technical change and allow for capital-augmenting technical change, point Z cannot be a steady state. To the right of κ^*, the growth rate of B_t

is strictly negative.

At the beginning of the transition, the elements of $\{\kappa_t\}_{t=2}^{\infty}$ are greater than κ^*, and those of $\{B_t\}_{t=2}^{\infty}$ are smaller than B^*, at least in $t = 2$.[9] The intuition for this provides Corollary 1. In period $t = 2$, the capital stock is predetermined, i. e., it will have grown at the steady-state rate such that $K_2 = K_1 \left(1 - \delta + g^A(\kappa^*)\right)(1+\lambda)$. However, the growth of the work force will have slowed down since $L_2 = L_1(1+\lambda') < L_1(1+\lambda)$. Therefore, there is capital-deepening and we have $K_2/L_2 > K_1/L_1$. Labor has become relatively scarcer than in the steady state. Accordingly, there is more labor-augmenting and less capital-augmenting technical progress than in the steady state, i. e., $\kappa_2 > \kappa^*$. Accordingly, the growth rate of capital-augmenting technological knowledge is negative since $g^B(\kappa_2) < \delta = g^B(\kappa^*)$. Hence, $B_2 < B^*$.

For how many periods κ_t will be increasing along the transition depends on particular shape of the difference equations (3.11) and (3.12). However, the following qualitative results can be derived.

Corollary 2 *Denote $\lambda_t \equiv L_t/L_{t-1} - 1$ the growth rate of the work force between period $t-1$ and t. Along the transition, if $\kappa_t > \kappa_{t-1}$, then*

$$\kappa_{t+1} > \kappa_t \quad \Leftrightarrow \quad \frac{1+\lambda_t}{1+\lambda_{t+1}} \left[1 - \delta + g^B(\kappa_t)\right] \frac{\tilde{\omega}(\kappa_t)}{\tilde{\omega}(\kappa_{t-1})} > 1. \tag{4.1}$$

According to Corollary 2, the condition for an evolution with $\kappa_{t+1} > \kappa_t$ can be given in terms of three factors. To grasp the intuition start with the case $\lambda_t = \lambda_{t+1}$. If $\kappa_t > \kappa_{t-1}$, then $\tilde{\omega}(\kappa_t) > \tilde{\omega}(\kappa_{t-1})$. Therefore, between $t-1$ and t, the real wage and individual savings will have grown faster than labor-augmenting technological knowledge. Moreover, under a constant population, aggregate savings and the capital stock will have grown faster than A_t/A_{t-1}. Therefore, if the growth factor of capital-augmenting technological knowledge is not too small, i. e., if $\left[1 - \delta + g^B(\kappa_t)\right] \tilde{\omega}(\kappa_t)/\tilde{\omega}(\kappa_{t-1}) > 1$, then $\kappa_{t+1} > \kappa_t$. As to Proposition 4, it is this latter condition that must hold if the transition has, e. g., $\kappa_3 > \kappa_2$, $\kappa_4 > \kappa_3$, etc. If the growth rate of the labor force is not constant along the transition, then a declining λ_t makes it easier to satisfy condition (4.1). Intuitively, this captures the effect of capital deepening between t and $t+1$.

An evolution involving increasing levels of κ_t has interesting implications for the evolution of other relevant magnitudes, too. On the one hand, labor productivity

[9] If the steady state is a stable node then the sequence $\{\kappa_t\}_{t=2}^{\infty}$ has some largest element and satisfies $\kappa_t > \kappa^*$ for all $t \geq 2$. Moreover, the sequence $\{B_t\}_{t=2}^{\infty}$ satisfies $B^* > B_t > B_{t+1}$ for all $t \geq 2$. If the steady state is a clockwise spiral sink, then the convergence of $\{\kappa_t, B_t\}_{t=2}^{\infty}$ to $(\kappa^{*\prime}, B^{*\prime})$ is oscillatory.

defined as $Y_t/L_t \equiv A_{t-1}\left(1-\delta+g^A\left(\kappa_t\right)\right) f\left(\kappa_t\right)$ increases in κ_t. This is due to the Solow effect through capital deepening and the Hicks effect on the incentives to engage in labor-augmenting technical change. On the other hand, capital productivity defined as $Y_t/K_t \equiv B_{t-1}\left(1-\delta+g^B\left(\kappa_t\right)\right) f\left(\kappa_t\right)/\kappa_t$ declines in κ_t. This is due to the diminishing marginal product of capital, the flip-side of the Solow effect, and the Hicks effect that reduces the incentives to engage in capital-augmenting technical change. Moreover, c_t^y and s_t increase in κ_t since the real wage does. Consumption of the old, c_t^o declines as a higher κ_t reduces the real interest rate at t.

5 Pensions and Growth

The question of this section is whether and how a pay-as-you-go pension scheme affects the direction of technical change. The central finding is such a scheme affects the relative scarcity of capital and labor through savings, i. e., an economy's capacity to accumulate capital.

Consider a government that runs a pay-as-you-go pension system with a balanced budget. Denote b_t the individual pension benefits at t and $\tau_t \in (0,1)$ the contribution rate at t. The budget constraint faced by the pension board is $b_t\, L_{t-1} = L_t \tau_t\, w_t$ such that

$$b_t = (1+\lambda)\,\tau_t\, w_t, \quad \text{for all } t = 1, 2, ..., \infty. \tag{5.1}$$

Denote the replacement rate by $\zeta \in (0,1)$. Then, the pension benefit when old is ζ times the net wage income earned by a current worker, i. e,

$$b_t = \zeta\, w_t\left(1-\tau_t\right). \tag{5.2}$$

The contribution rate is endogenous and adjusts such that the budget of the pension scheme is balanced. Upon combining (5.1) and (5.2), this requires $\tau_t = \hat{\tau}$, where

$$\hat{\tau} = \frac{\zeta}{\zeta+1+\lambda} \equiv \hat{\tau}\left(\zeta, \lambda\right), \quad \text{with} \quad \frac{\partial \hat{\tau}}{\partial \zeta} > 0 \quad \text{and} \quad \frac{\partial \hat{\tau}}{\partial \lambda} < 0. \tag{5.3}$$

As long as the replacement and the population growth rate remain constant, so does the contribution rate. An increase in the replacement rate increases benefits. To counteract this rise and to raise contributions, the contribution rate must increase. A decline in the population growth rate increases the ratio of retirees to contributors such that a given replacement rate must be supported by a higher contribution rate.

The household's problem of maximizing (2.1) is now subject to the per-period budget constraints $c_t^y + s_t = (1-\hat{\tau})w_t$ and $c_{t+1}^o = s_t(1+r_{t+1}) + b_{t+1}$. Under log-utility, optimal savings at t result as

$$s_t = \frac{\beta}{1+\beta}(1-\hat{\tau})w_t - \frac{1}{1+\beta}\left(\frac{b_{t+1}}{1+r_{t+1}}\right), \tag{5.4}$$

which extends (2.4) to the case $\hat{\tau} > 0$.

Suppose the pension scheme described by (5.1) and (5.2) is introduced in period $t = 1$ and maintained thereafter. Then, the old of generation $t = 0$ are the first to receive a benefit and the young of generation $t = 1$ are the first contributors. The equilibrium with a pension scheme determines the same sequences as stated in Section 3.1 with individual savings given by (5.4). Throughout, I assume that ζ is sufficiently small such that $s_t > 0$ along the transition and in the steady state. For a given government policy described by $(\zeta, \hat{\tau}, b_t)_{t=1}^{\infty}$ and initial values $(K_1, L_1, A_0, B_0) > 0$ conditions (E1)-(E7) determine a sequence of state variables $\{\kappa_t, B_t\}_{t=1}^{\infty}$.

Proposition 5 *(Dynamical System and Steady State with Pension Scheme)*

Denote

$$\tilde{\beta} \equiv \frac{\beta(1-\hat{\tau})}{(1+\beta)(1+\lambda)}, \quad \tilde{G}(\kappa_t) \equiv \frac{1-\delta+g^A(\kappa_t)}{1-\delta+g^B(\kappa_t)}\left[\kappa_t + \frac{\hat{\tau}}{1+\beta}\frac{\tilde{w}(\kappa_t)}{\tilde{R}(\kappa_t)}\right]. \tag{5.5}$$

1. *There is a unique sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$ determined by*

$$\tilde{\beta}\tilde{w}(\kappa_t) = \frac{\tilde{G}(\kappa_{t+1})}{B_t}, \tag{5.6}$$

   and (3.12), where κ_1 and B_1 satisfy (3.13).

2. *If (3.16) holds, then the dynamical system with a pension scheme has a unique steady state involving $\kappa_\zeta^* \in (0,\infty)$ and $B_\zeta^* \in (0,\infty)$. Moreover, it holds that*

$$\kappa_\zeta^* = \kappa^* \quad \text{and} \quad B_\zeta^* > B^*. \tag{5.7}$$

Hence, the evolution of the economy with a pension scheme can be studied through the lens of the sequence of the same state variables as before. The new steady state has the same efficient capital intensity, $\kappa_\zeta^* = \kappa^*$, which is determined by technology alone. As a consequence, the steady-state growth rate does not depend on the pension scheme either. However, there is a positive level effect of $\hat{\tau}$ on B^*. To understand why $B_\zeta^* > B^*$, it is useful to study the transitional dynamics induced by the pension scheme.

Proposition 6 *(Local Stability and Transitional Dynamics)*

1. *If the locus* $\Delta\kappa_t = 0$ *is stable in the vicinity of* $\left(\kappa_\zeta^*, B_\zeta^*\right)$*, then this steady state is either a stable node or a clockwise spiral sink.*

2. *Suppose the initial values* $(K_1, L_1, A_0, B_0) > 0$ *are such that* $\kappa_1 = \kappa^*$ *and* $B_1 = B^*$*. Then, the establishment of a pension scheme involving (5.1) and (5.3) in* $t = 1$ *leads to* $\kappa_2 < \kappa^*$ *and* $B_2 > B^*$*.*

3. *Let* ζ *be such that* (κ_2, B_2) *is in a sufficiently small neighborhood of* $\left(\kappa_\zeta^*, B_\zeta^*\right)$*. Then, the economy embarks on a trajectory* $\{\kappa_t, B_t\}_{t=2}^{\infty}$ *with* $\lim_{t\to\infty}\{\kappa_t, B_t\} = \left(\kappa_\zeta^*, B_\zeta^*\right)$*.*

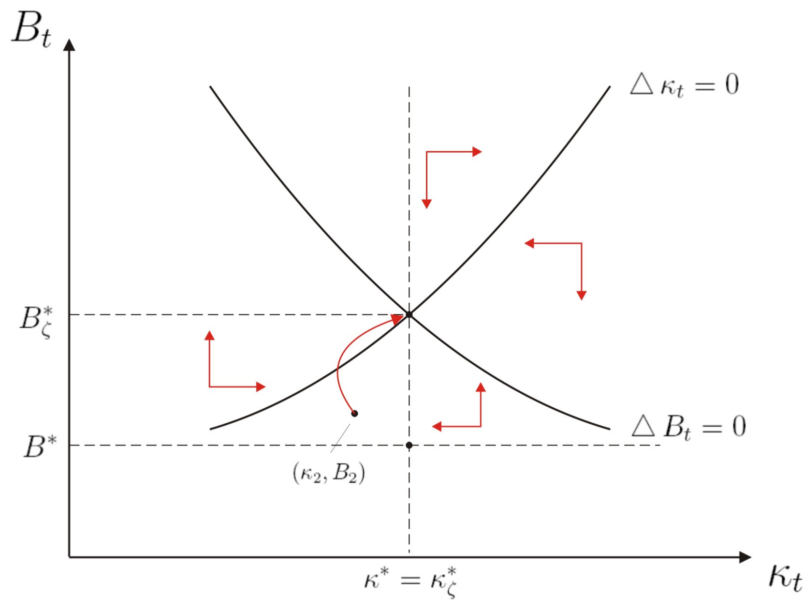
Hence, if the process of capital accumulation is locally stable for a time-invariant $B_t = B$, then the pension scheme does not alter the local stability properties of the steady state. According to the second part of Proposition 6, the introduction of the pension scheme initially reduces κ_t. Indeed, ceteris paribus, due to the pension scheme the net wage income when young falls and the income when old increases. Hence, the incentives to save decline for all generations $t \geq 1$. Therefore, K_2 is smaller with a pension scheme than without. According to Corollary 1, $\kappa_2 < \kappa^*$. As a consequence, there is initially more capital-augmenting technical change and less labor-augmenting technical change than at (κ^*, B^*). Therefore, $B_2 < B^*$. For $t > 2$, the evolution of the state variables depends on the stability properties of $\left(\kappa_\zeta^*, B_\zeta^*\right)$. Figure 5.1 shows the case of a stable node. Here, B_t grows at a strictly positive rate with $\lim_{t\to\infty} B_t = B_\zeta^* > B^*$.

6 Population, Pensions, and Growth

The purpose of this section is to analyze the interaction between the pension scheme, demographic changes, and the direction of technical change. The central result is that this interaction depends on the institutional characteristics of the pension scheme. To highlight this point, we compare the effects of a permanent decline in the population growth rate under a pay-as-you-go pension scheme with a constant replacement rate to the effects if the pension scheme keeps the contribution rate constant. The following proposition has the results under a constant replacement rate.

Proposition 7 *(Population, Pensions, and Growth)*

Figure 5.1: Comparative Statics and Dynamics following the Introduction of a Pension Scheme. The Case of a Stable Node.



Consider an economy in the steady state $\left(\kappa_\zeta^*, B_\zeta^*\right)$ *in period* $t = 1$*. Then, it experiences a small and permanent decline in the growth rate of its labor force such that* $L_t = L_1(1 + \lambda')$ *with* $\lambda > \lambda' > (-1)$ *for all* $t = 2, 3, ..., \infty$*. Denote* $\left(\kappa_\zeta^{*\prime}, B_\zeta^{*\prime}\right)$ *the steady state associated with* λ'*.*

1. *It holds that* $\kappa_\zeta^{*\prime} = \kappa_\zeta^*$ *and* $B_\zeta^{*\prime} < B_\zeta^*$
2. *It holds that* $\kappa_2 > \kappa_\zeta^*$ *and* $B_2 < B_\zeta^*$*. If* λ' *is such that* (κ_2, B_2) *is in a sufficiently small neighborhood of* $\left(\kappa_\zeta^{*\prime}, B_\zeta^{*\prime}\right)$*, then the economy embarks on a trajectory* $\{\kappa_t, B_t\}_{t=2}^{\infty}$ *with* $\lim_{t\to\infty} \{\kappa_t, B_t\} = \left(\kappa_\zeta^{*\prime}, B_\tau^{*\prime}\right)$*.*

At first sight, the presence of a pension scheme with a constant replacement rate does not affect the qualitative results of Proposition 4 where I analyze a permanent decline in the population growth rate in an economy without a pension scheme. However, here a major new facet is that the decline in the population growth rate requires a higher contribution rate to the pension scheme. This reduces both the net income when young and the pension benefit when old. While the former effect decreases individual savings, the latter increases it. As to the steady state, the net effect is positive: individuals save more to make up for the lower expected pension benefit. Therefore, $B_\zeta^{*'} < B_\zeta^*$.

More savings also explain why $\kappa_2 > \kappa_\zeta^*$. Intuitively, generation $t = 1$ anticipates a lower pension benefit for two reasons. First, there are fewer contributors to the system, and, second, the contribution rate increases, which reduces the net wage to which the replacement rate applies. To make up for this, individuals increase savings. As a result, between $t = 1$ and $t = 2$, the capital stock grows faster due to the decline in the growth rate of the labor force. At the same time, labor force growth slows down. As the result, there is capital deepening, i. e., $K_2/L_2 > K_1/L_1$ and faster labor- and slower capital-augmenting technical change. Hence, the economy embarks on a trajectory where initially the real wage grows faster than labor productivity because of the Solow effect and the Hicks effect on labor-augmenting technical change.

Does the same conclusion hold true if we change the institutional characteristics of the pension scheme? To address this questions, consider a pension scheme with a constant contribution rate instead of a constant replacement rate, i. e., replace (5.1) by $b_t = (1 + \lambda)\,\tau\, w_t$ and (5.4) by

$$s_t = \frac{\beta}{1+\beta}(1-\tau)w_t - \frac{1}{1+\beta}\left(\frac{(1+\lambda)\,\tau\, w_{t+1}}{1+r_{t+1}}\right), \tag{6.1}$$

where $\tau \in (0, 1)$ is the constant contribution rate.

It is straightforward to establish the same qualitative results as stated in Proposition 5 and Proposition 6 for the economy with a pension scheme. Roughly speaking, introducing a pension scheme is equivalent to a hike in τ. From Proposition 5 such a change leaves the steady-state efficient capital intensity unaffected. Moreover, since, ceteris paribus, an increase in τ reduces current disposable income and increases the expected pension benefit, for all cohorts $t = 1, 2, ...$ the incentive to save declines in response to an increase in τ at $t = 1$. Therefore, K_2 will be smaller than without the increase in the contribution rate. As a consequence, at the beginning of the transition there is capital widening in the sense that $K_1/L_1 > K_2/L_2$, capital becomes relatively scarcer, and the incentives to engage in capital-augmenting (labor-augmenting) technical change increase (decrease). In fact, if the steady satte is a stable node we have along the transition $g^B(\kappa_t) > 0 = g^B(\kappa^*)$.

In spite of these similarities, the next result shows that the pension scheme affects the transition following a decline in the growth rate of the labor force in differential ways.

Proposition 8 *(Comparative Institutional Analysis)*

Consider two economies that differ only with respect to their pension scheme. The first economy runs its pension scheme with a constant replacement rate, the second *a scheme with a constant contribution rate. Suppose that both economies are in a steady state in* $t = 1$ *with* $\left(\kappa_{\zeta}^{*}, B_{\zeta}^{*}\right) = \left(\kappa_{\tau}^{*}, B_{\tau}^{*}\right)$*, i. e.,* $\tau = \hat{\tau}(\zeta, \lambda)$*. Then, these economies experience a small and permanent decline in the growth rate of their labor force such that* $L_t = L_1(1+\lambda')^{t-1}$ *with* $\lambda > \lambda' > (-1)$*.*

1. *It holds that* $\kappa_{\zeta}^{*} = \kappa_{\tau}^{*} = \kappa_{\zeta}^{*\prime} = \kappa_{\tau}^{*\prime}$*,* $B_{\zeta}^{*} = B_{\tau}^{*} > B_{\zeta}^{*\prime} > B_{\tau}^{*\prime}$*.*

2. *Suppose that* λ' *is such that* $(\kappa_2, B_2)|_{\zeta=const.}$ *is in a sufficiently small neighborhood of* $\left(\kappa_{\zeta}^{*\prime}, B_{\zeta}^{*\prime}\right)$ *and* $(\kappa_2, B_2)|_{\tau=const.}$ *is in a sufficiently small neighborhood of* $(\kappa_{\tau}^{*\prime}, B_{\tau}^{*\prime})$*. Then, these economies embark on converging trajectories. Moreover, it holds that*

$$\kappa_2|_{\zeta=const.} < \kappa_2|_{\tau=const.} \quad \text{and} \quad B_2|_{\zeta=const.} > B_2|_{\tau=const.}. \tag{6.2}$$

According to the first part of Proposition 8, a permanent decline in the growth rate of the labor force generates no differential growth effects in the long run. The steady-state level effect on capital-augmenting technical knowledge is due to the differential way in which the pension system affects the incentives to save. Under both systems, individuals save more knowing that fewer workers contribute to the system. In addition, if the replacement rate is constant, then a decline in the growth rate of the labor force growth leads to an increase in the contribution rate. this has two opposing effects on the incentive to save. On the one hand, current disposable income declines, on the other hand the expected pension benefit declines.

What matters for the second claim is that generation $t = 1$ adjusts its savings under rational expectations. Under a constant contribution rate, they expect a greater reduction in the pension benefits and accordingly save more. Therefore K_2 is greater under a constant contribution rate. Accordingly, capital deepening is more pronounced and leads to more labor- and less capital saving technical change.

7 Concluding Remarks

The present paper is about changing factor endowments and the direction of technical change. It takes the demographic evolution as exogenous studies the economic consequences for capital accumulation, capital- and labor-augmenting technical change.

to be completed

A Proofs

A.1 Proof of Proposition 1

Upon substitution of (2.8) and (2.9) in the respective zero-profit condition of (2.17), we obtain

$$\begin{aligned} f(\kappa_t) - \kappa_t f'(\kappa_t) &= (1-\delta+\hat{q}_t^A) i'(\hat{q}_t^A) + i(\hat{q}_t^A), \\ f'(\kappa_t) &= (1-\delta+\hat{q}_t^B) i'(\hat{q}_t^B) + i(\hat{q}_t^B). \end{aligned} \tag{A.1}$$

Denote $RHS(q)$ the right-hand side of both conditions. In view of the properties of the function i given in (2.12), $RHS(q)$ is a mapping $RHS(q): \mathbb{R}_+ \to \mathbb{R}_+$ with $RHS'(q) > 0$ for $q \geq 0$ and $\lim_{q\to\infty} RHS(q) = \infty$. Moreover, the properties of the function $f(\kappa_t)$ imply that the left-hand side of both conditions is strictly positive for $\kappa_t > 0$. Hence, for each $\kappa_t > 0$ there is a unique $\hat{q}_t^j > 0$, $j = A, B$, that satisfies the respective condition stated in (A.1). I denote these maps by $\hat{q}_t^j = g^j(\kappa_t)$, $j = A, B$.

An application of the implicit function theorem gives

$$\begin{aligned} \frac{d\hat{q}^A}{d\kappa_t} &= \frac{-\kappa_t f''(\kappa_t)}{\left(1-\delta+\hat{q}_t^A\right) i''\left(\hat{q}_t^A\right) + 2i'\left(\hat{q}_t^A\right)} \equiv g_\kappa^A(\kappa_t) > 0, \\ \frac{d\hat{q}^B}{d\kappa_t} &= \frac{f''(\kappa_t)}{\left(1-\delta+\hat{q}_t^B\right) i''\left(\hat{q}_t^B\right) + 2i'\left(\hat{q}_t^B\right)} \equiv g_\kappa^B(\kappa_t) < 0. \end{aligned} \tag{A.2}$$

The respective signs follow from the properties of the functions f and $i(q_t^j)$. ■

A.2 Proof of Corollary 1

Equation (3.8) is a fixed-point problem with a unique solution $\hat{\kappa}_t > 0$. To see this, write the right-hand side of (3.8) as $RHS(\kappa_t, K_t/L_t, A_{t-1}/B_{t-1})$. Given the properties of $g^A(\kappa_t)$ and $g^B(\kappa_t)$ as stated in (2.18), $RHS(\kappa_t, K_t/L_t, A_{t-1}/B_{t-1})$ is continuous and strictly decreasing for all $\kappa_t > 0$. Moreover, it is strictly positive since $K_t/L_t > 0$ and $A_{t-1}/B_{t-1} > 0$, and for κ_t near 0 we have $RHS(\kappa_t, K_t/L_t, A_{t-1}/B_{t-1}) > \kappa_t$. Hence, there is a unique fixed-point $\hat{\kappa}_t > 0$.

Ceteris paribus, a higher K_t/L_t shifts the function $RHS(\kappa_t, K_t/L_t, A_{t-1}/B_{t-1})$ upwards. Therefore, $\hat{\kappa}_t$ is greater the greater K_t/L_t. Then, from (2.18), $\hat{q}_t^A = g^A(\kappa_t)$ increases whereas $\hat{q}_t^B = g^B(\kappa_t)$ decreases. Obviously, the opposite response obtains when K_t/L_t falls. ■

A.3 Proof of Proposition 2

The proof consists of two steps. First, I show that the variables κ_t and B_t are indeed state variables of the economy at t. Second, I prove the existence of a unique sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$.

1. At t the equilibrium determines 25 variables. One readily verifies that there are also 25 conditions for each period. Since $\hat{q}^A = g^A(\kappa_t)$ and $\hat{q}^B = g^B(\kappa_t)$ according to Proposition 1, it is also straightforward to verify that all prices $\{p_{L,t}, p_{K,t}, r_t, w_t, R_t\}_{t=1}^{\infty}$ depend on κ_t. Moreover, $\{Y_t, Y_{K,t}, Y_{L,t}, n_t, m_t, a_t, b_t, l_t, k_t\}_{t=1}^{\infty}$ as well as $\{A_t, B_t\}_{t=1}^{\infty}$ depend on κ_t. In addition, $y_{L,t} = y_{K,t} = 1$. With this in mind, individual savings, (2.3), become $s_t = s[w(\kappa_t), R(\kappa_{t+1})]$, where $w(\kappa_t) \equiv A_{t-1}\left(1-\delta+g^A(\kappa_t)\right)\left[f(\kappa_t)-\kappa_t f'(\kappa_t) - i\left(g^A(\kappa_t)\right)\right]$ and $R(\kappa_{t+1}) = 1 + r(\kappa_{t+1}) \equiv B_t\left(1-\delta+g^B(\kappa_{t+1})\right)\left[f'(\kappa_{t+1}) - i\left(g^B(\kappa_{t+1})\right)\right]$. Both factor prices, $w(\kappa_t)$ and $R(\kappa_{t+1})$, are found from the respective zero-profit condition given in (2.13) and Proposition 1. The individual budget constraints deliver $c_t^y = w(\kappa_t) - s[w(\kappa_t), R(\kappa_{t+1})]$ and $c_t^o = R(\kappa_t)\, s[w(\kappa_{t-1}), R(\kappa_t)]$. For $t=1$, we have by assumption that $c_1^o = R(\kappa_1) K_1/L_0$. Under the assumption of log-utility the expressions for c_t^y, s_t, and c_t^o simplify in a straightforward fashion.

2. First, I derive the difference equation (3.11). Consider the equation for capital accumulation of (3.5) and use the results established under 1. This gives

$$s[w(\kappa_t), R(\kappa_{t+1})] = \kappa_{t+1}(1+\lambda)\frac{A_{t+1}}{B_{t+1}}, \quad \text{for } t = 1, 2, ..., \infty. \tag{A.3}$$

In this general setting the dynamical system would comprise three difference equations, namely (A.3) as well as from (2.20)

$$A_{t+1} = A_t\left(1-\delta+g^A(\kappa_{t+1})\right) \quad \text{and} \quad B_{t+1} = B_t\left(1-\delta+g^B(\kappa_{t+1})\right). \tag{A.4}$$

The assumption of log-utility implies that $s[w(\kappa_t), R(\kappa_{t+1})] = \beta/(1+\beta)w(\kappa_t)$. Since w_t is proportionate to A_t, we may rewrite (A.3) as

$$\frac{\beta}{1+\beta}\tilde{w}(\kappa_t) = \kappa_{t+1}(1+\lambda)\frac{1-\delta+g^A(\kappa_{t+1})}{B_t\left(1-\delta+g^B(\kappa_{t+1})\right)}, \tag{A.5}$$

which no longer depends on the state A_{t+1} but on the growth factor A_{t+1}/A_t. Hence, under log-utility the number of state variables reduces to two. Obviously, (A.5) coincides with (3.11) if we use the definition of $G(\kappa_t)$ given in (3.10). The second difference equation of the dynamical system is the one for B_{t+1} given in (A.4) which restates (3.12).

In the first period, κ_1 is not pinned down by equilibrium conditions, however, K_1 is given. Then, κ_1 is determined by its definition using Proposition 1. From the proof of Corollary 1, the resulting fixed-point problem has a unique solution $\kappa_1 > 0$.

Before we turn to the uniqueness of the sequence (κ_t, B_t) it proves useful to state an prove the following lemma.

Lemma 2 *It holds for all $\kappa_t > 0$ that*

$$\begin{aligned} \tilde{w}(\kappa_t) &> 0 \quad \textit{and} \quad \tilde{w}'(\kappa_t) > 0, \\ \tilde{R}(\kappa_t) &> 0 \quad \textit{and} \quad \tilde{R}'(\kappa_t) < 0, \\ G(\kappa_t) &> 0 \quad \textit{and} \quad G'(\kappa_t) > 0 \quad \textit{with} \quad \lim_{\kappa_t \to \infty} G(\kappa_t) = \infty. \end{aligned} \tag{A.6}$$

Proof of Lemma 2

First, I note that (2.13), Proposition 1, and the updating conditions (2.20) deliver

$$\begin{aligned} w_t/A_t &= f(\kappa_t) - \kappa_t f'(\kappa_t) - i\left(g^A(\kappa_t)\right) \equiv \tilde{w}(\kappa_t), \\ R_t/B_t &= f'(\kappa_t) - i\left(g^B(\kappa_t)\right) \equiv \tilde{R}(\kappa_t). \end{aligned} \tag{A.7}$$

With (A.1), this gives

$$\begin{aligned} \tilde{w}(\kappa_t) &= \left(1-\delta+g^A(\kappa_t)\right) i'\left(g^A(\kappa_t)\right) > 0, \\ \tilde{R}(\kappa_t) &= \left(1-\delta+g^B(\kappa_t)\right) i'\left(g^B(\kappa_t)\right) > 0, \end{aligned} \tag{A.8}$$

where, for $\kappa_t > 0$, the signs follow from Proposition 1 and the properties of the function i given in (2.12). It follows that

$$\begin{aligned} \tilde{w}'(\kappa_t) &= g_\kappa^A(\kappa_t)\left[i'\left(g^A(\kappa_t) + \left(1-\delta+g^A(\kappa_t)\right) i''\left(g^A(\kappa_t)\right)\right)\right] > 0, \\ \tilde{R}'(\kappa_t) &= g_\kappa^B(\kappa_t)\left[i'\left(g^B(\kappa_t) + \left(1-\delta+g^B(\kappa_t)\right) i''\left(g^B(\kappa_t)\right)\right)\right] < 0. \end{aligned} \tag{A.9}$$

Again, for $\kappa_t > 0$, the signs follow from Proposition 1 and the properties of the function i given in (2.12).

The third claim of (A.6) is immediate from the definition of G given in (3.10), and Proposition 1. ■

According to Lemma 2, $\tilde{w}(\kappa_t)$ and $G(\kappa_t)$ are strictly increasing functions on $\mathbb{R}_{++}$. Therefore, given $(\kappa_t, B_t) \in \mathbb{R}_{++}$, there is a unique $(\kappa_{t+1}, B_{t+1}) \in \mathbb{R}_{++}$ that satisfies (3.11) and (3.12). Intuitively, given $(\kappa_t, B_t) \in \mathbb{R}_{++}$, there is a unique $\kappa_{t+1} \in \mathbb{R}_{++}$ that satisfies (3.11) since $G'(\kappa_t) > 0$ and $\lim_{\kappa_t \to \infty} G(\kappa_t) = \infty$. With this value at hand, (3.12) delivers a unique $B_{t+1} > 0$. ■

A.4 Proof of Proposition 3

1. Condition (3.16) is necessary and sufficient for (3.15) to have at least one solution $\kappa^* \in (0, \infty)$ given that $g^B(\kappa)$ is continuous according to Proposition 1. Then, existence of a steady state $(\kappa^*, B^*) \in \mathbb{R}_{++}$ follows from (3.14) and Lemma 2. They assure that a value $B^* = (1+\beta)(1+\lambda)\, G(\kappa^*) / (\beta\, \tilde{w}(\kappa^*)) \in (0, \infty)$ exists since $G(\kappa) > 0$ and $\tilde{w}(\kappa) > 0$ for any $\kappa \in (0, \infty)$. Uniqueness follows from $g_\kappa^B(\kappa) < 0$ on $\mathbb{R}_{++}$.
2. From (2.8), we have $p_K^* = f'(\kappa^*)$. Then, (A.7) delivers $R^* = B^*[p_K^* - i(\delta)]$. With (E6) the steady-state real interest rate r^* results. The stated findings about the steady-state growth rate of w_t, c_t^y, c_o^y, and s_t are immediate from (A.7), (2.4), and the per-period budget constraints of each individual. ■

A.5 Proof of Lemma 1

The following Result introduces some necessary notation for the analysis of the local stability of the steady state.

Result 1 *With* (κ_1, B_1) *given by (3.13), the dynamical system can be stated as*

$$(\kappa_{t+1}, B_{t+1}) = \phi(\kappa_t, B_t) \equiv \left(\phi^{\kappa}(\kappa_t, B_t), \phi^{B}(\kappa_t, B_t)\right), \; t = 1, 2, ..., \infty, \tag{A.10}$$

where $\phi^j : \mathbb{R}_{++} \to \mathbb{R}_{++},\; j = \kappa, B,$ *are continuously differentiable functions.*

Proof of Result 1

Write (3.11) and (3.12) as $\kappa_{t+1} = h(\kappa_t, B_{t+1})$ and $B_{t+1} = z(\kappa_{t+1}, B_t)$, where h and z are some continuously differentiable functions. With B_0 as an initial condition and κ_1 determined by (3.13), B_1 is the equal to $z(B_0, \kappa_1)$. Hence, we may state the the dynamical system as

$$\kappa_{t+1} = h(\kappa_t, B_{t+1}), \quad B_{t+1} = z(\kappa_{t+1}, B_t) \qquad t = 1, 2, ..., \infty, \; (\kappa_1, B_1) \text{ given.} \tag{A.11}$$

Then, for any (κ_t, B_t) these equations determine (κ_{t+1}, B_{t+1}). More precisely, upon substitution we obtain $\kappa_{t+1} = h(\kappa_t, z(\kappa_{t+1}, B_t))$ which implicitly defines $\kappa_{t+1} = \phi^{\kappa}(\kappa_t, B_t)$. In turn, using the latter, we find $B_{t+1} = z(\kappa_{t+1}, B_t) = z(\phi^{\kappa}(\kappa_t, B_t), B_t) \equiv \phi^{B}(\kappa_t, B_t)$. Hence, we may state the dynamical system as in (A.10). ■

The steady state is a fixed point of the system (A.10). To study the local behavior of the system around the steady state, we have to know the eigenvalues of the Jacobian matrix

$$D\phi(\kappa^*, B^*) \quad \equiv \quad \begin{pmatrix} \phi^{\kappa}_{\kappa}(\kappa^*, B^*) & \phi^{\kappa}_{B}(\kappa^*, B^*) \\ \phi^{B}_{\kappa}(\kappa^*, B^*) & \phi^{B}_{B}(\kappa^*, B^*) \end{pmatrix}. \tag{A.12}$$

We study each of the four elements of the Jacobian in turn.

1. First, consider $\phi^{\kappa}(\kappa_t, B_t)$. Consider (3.11), which we repeat here for convenience introducing $\hat{\beta}$ for brevity

   $$\hat{\beta}\, \tilde{w}(\kappa_t) = \frac{G(\kappa_{t+1})}{B_t}.$$

   The implicit function theorem assures that a function $\kappa_{t+1} = \phi^{\kappa}(\kappa_t, B_t)$ exists if $\phi^{\kappa}_{\kappa}(\kappa_t, B_t) \equiv d\kappa_{t+1}/d\kappa_t$ and $\phi^{\kappa}_{B}(\kappa_t, B_t) \equiv d\kappa_{t+1}/dB_t$ exist. We show next that this is the case for all $(\kappa_t, B_t) > 0$.

   (a) We start with $\phi^{\kappa}_{\kappa}(\kappa_t, B_t)$. Implicit differentiation of (3.11) gives

   $$\phi^{\kappa}_{\kappa}(\kappa_t, B_t) \quad = \quad \frac{\hat{\beta}\, B_t\, \tilde{w}'(\kappa_t)}{G'(\kappa_{t+1})} > 0. \tag{A.13}$$

   Form Lemma 2 the numerator and the denominator are strictly positive. Hence, the derivative $\phi^{\kappa}(\kappa_t, B_t)$ exists and is strictly positive for all (κ_t, B_t).

   Evaluated at the steady state, (A.13) simplifies to

   $$\phi^{\kappa}_{\kappa}(\kappa^*, B^*) \quad = \quad \frac{G(\kappa^*)\, w'(\kappa^*)}{G'(\kappa^*)\, w(\kappa^*)}. \tag{A.14}$$

The assumption that the locus $\Delta\kappa_t = 0$ is stable in the vicinity of the steady state is equivalent to $\phi^{\kappa}_{\kappa}(\kappa^*, B^*) \in (0,1)$. By (A.13) we have $\phi^{\kappa}_{\kappa}(\kappa^*, B^*) > 0$. Moreover, using the definition of G given in (3.10) as well as Lemma 2, I find

$$\phi^{\kappa}_{\kappa}(\kappa^*, B^*) < 1 \quad \Leftrightarrow \quad i''\left(g^A(\kappa^*)\right) g^A_{\kappa}(\kappa^*) \frac{\kappa^*}{i'\left(g^A(\kappa^*)\right)} < 1 - g^B_{\kappa}(\kappa^*). \tag{A.15}$$

Hence, the assumption that $\phi^{\kappa}_{\kappa}(\kappa^*, B^*) \in (0,1)$ means that (A.15) is satisfied at κ^*. Roughly speaking, this condition says that the elasticity of $i'\left(g^A(\kappa^*)\right)$ with respect to κ must not be too large.

(b) Next, we turn to the derivative $\phi^{\kappa}_{B}(\kappa_t, B_t)$. Total differentiation of (3.11) gives now

$$\phi^{\kappa}_{B}(\kappa_t, B_t) \;=\; \frac{G\left(\kappa_{t+1}\right)}{B_t\, G'\left(\kappa_{t+1}\right)} > 0. \tag{A.16}$$

By Lemma 2, the derivative exists for all $\kappa_{t+1} > 0$ and is strictly positive. Evaluated at the steady state, we have

$$\phi^{\kappa}_{B}(\kappa^*, B^*) \;=\; \frac{G\left(\kappa^*\right)}{B^*\, G'\left(\kappa^*\right)}. \tag{A.17}$$

2. Next, we turn to $\phi^B\left(\kappa_t, B_t\right)$. Consider (3.12) for $t+1$ and substitute $\kappa_{t+1} = \phi^{\kappa}\left(\kappa_t, B_t\right)$. This gives

$$B_{t+1} = B_t\left(1 - \delta + g^B\left(\kappa_{t+1}\right)\right) = B_t\left(1 - \delta + g^B\left(\phi^{\kappa}\left(\kappa_t, B_t\right)\right)\right) \equiv \phi^B\left(\kappa_t, B_t\right). \tag{A.18}$$

Since $\phi^{\kappa}\left(\kappa_t, B_t\right)$ exists for all $\left(\kappa_t, B_t\right) > 0$ so does $\phi^B\left(\kappa_t, B_t\right)$. We now characterize the partial derivatives of this function.

(a) First, consider $\phi^B_{\kappa}\left(\kappa_t, B_t\right) \equiv \partial B_{t+1}/\partial\kappa_t$. From (A.18) we have

$$\phi^B_{\kappa}\left(\kappa_t, B_t\right) \;=\; B_t\, g^B_{\kappa}\left(\phi^{\kappa}\left(\kappa_t, B_t\right)\right)\, \phi^{\kappa}_{\kappa}\left(\kappa_t, B_t\right) < 0. \tag{A.19}$$

Evaluated at the steady state, this gives

$$\phi^B_{\kappa}\left(\kappa^*, B^*\right) \;=\; B^*\, g^B_{\kappa}\left(\kappa^*\right)\, \phi^{\kappa}_{\kappa}\left(\kappa^*, B^*\right). \tag{A.20}$$

(b) Next, we consider the derivative $\phi^B_B\left(\kappa_t, B_t\right) \equiv \partial B_{t+1}/\partial B_t$. From (A.18), we find

$$\phi^B_B\left(\kappa_t, B_t\right) \;=\; \left(1 - \delta + g^B\left(\phi^{\kappa}\left(\kappa_t, B_t\right)\right)\right) + B_t\, g^B_{\kappa}\left(\phi^{\kappa}\left(\kappa_t, B_t\right)\right)\, \phi^{\kappa}_B\left(\kappa_t, B_t\right).$$

Evaluated at the steady state, this gives

$$\phi^B_B\left(\kappa^*, B^*\right) \;=\; 1 + B^*\, g^B_{\kappa}\left(\kappa^*\right)\, \phi^{\kappa}_B\left(\kappa^*, B^*\right). \tag{A.21}$$

Observe that $\phi^B_B\left(\kappa^*, B^*\right) \in (0,1)$. Indeed, since $g^B_{\kappa} < 0$, we have $\phi^B_B\left(\kappa^*, B^*\right) < 1$. Using (A.17) the definition of G of (3.10) I find

$$\phi^B_B\left(\kappa^*, B^*\right) > 0 \quad \Leftrightarrow \quad 1 - \delta + g^A(\kappa^*) + \kappa^*\, g^A_{\kappa}(\kappa^*) > 0, \tag{A.22}$$

where the sign follows from Proposition 1.

Using the results of (A.14), (A.17), (A.20), and (A.21), the required Jacobian (A.12) can be written

$$D\,\phi(\kappa^*, B^*) = \begin{pmatrix} \phi^\kappa_\kappa(\kappa^*, B^*) & \phi^\kappa_B(\kappa^*, B^*) \\ B^* g^B_\kappa\,(\kappa^*)\ \phi^\kappa_\kappa(\kappa^*, B^*) & 1 + B^*\, g^B_\kappa\,(\kappa^*)\ \phi^\kappa_B(\kappa^*, B^*) \end{pmatrix}. \tag{A.23}$$

Denote μ_1 and μ_2 the eigenvalues of the Jacobian (A.23). They satisfy the characteristic equation at (κ^*, B^*), i. e.,

$$\mu_{1,2} = \frac{tr\,(D\phi)}{2} \pm \sqrt{\left(\frac{tr\,(D\phi)}{2}\right)^2 - det\,(D\phi)}. \tag{A.24}$$

With (A.14), (A.17), (A.20), and (A.21), we find

$$\begin{aligned} \mu_1 &= \frac{\phi^\kappa_\kappa + \phi^B_B}{2} + \sqrt{\left(\frac{\phi^\kappa_\kappa + \phi^B_B}{2}\right)^2 - \phi^\kappa_\kappa}, \\ \mu_2 &= \frac{\phi^\kappa_\kappa + \phi^B_B}{2} - \sqrt{\left(\frac{\phi^\kappa_\kappa + \phi^B_B}{2}\right)^2 - \phi^\kappa_\kappa}. \end{aligned} \tag{A.25}$$

Both eigenvalues are real if $\phi^B_B \geq 2\sqrt{\phi^\kappa_\kappa} - \phi^\kappa_\kappa \geq 0$, or

$$1 + B^*\, g^B_\kappa\,(\kappa^*)\ \phi^\kappa_B(\kappa^*, B^*) \geq 2\sqrt{\phi^\kappa_\kappa} - \phi^\kappa_\kappa. \tag{A.26}$$

One readily verifies that μ_1 is strictly increasing in ϕ^B_B with $\mu_1|_{\phi^B_B=1} = 1$. Moreover, $\mu_1|_{\phi^B_B=2\sqrt{\phi^\kappa_\kappa}-\phi^\kappa_\kappa} = \sqrt{\phi^\kappa_\kappa}$. Hence, $\mu_1 \in (0,1)$. On the other hand, μ_2 decreases in ϕ^B_B with $\mu_2|_{\phi^B_B=2\sqrt{\phi^\kappa_\kappa}-\phi^\kappa_\kappa} = \sqrt{\phi^\kappa_\kappa}$ and $\mu_2|_{\phi^B_B} = \phi^\kappa_\kappa$. Hence, $\mu_2 \in (0, \sqrt{\phi^\kappa_\kappa})$. Moreover, $\mu_1 \geq \mu_2$ with equality when $\phi^B_B = 2\sqrt{\phi^\kappa_\kappa} - \phi^\kappa_\kappa$.

If (A.26) is violated, then $D\phi$ has two distinct complex eigenvalues. Then, the steady state is a spiral sink since $det\,(D\phi) = \phi^\kappa_\kappa < 1$ (see, e. g., Galor (2007), Proposition 3.8). The stability of the loci $\Delta\kappa_t = 0$ and $\Delta B_t = 0$ imply the clockwise orientation of the spiral sink. ■

A.6 Proof of Proposition 4

1. This follows immediately from (3.14) and (3.15) for λ and λ' and $\lambda > \lambda'$.
2. This follows since $(\kappa^{*\prime}, B^{*\prime})$ is a locally stable node or a clockwise spiral sink. Moreover, λ' is such that (κ_2, B_2) is in a sufficiently small neighborhood of $(\kappa^{*\prime}, B^{*\prime})$. Since K_2 is predetermined in $t = 2$, we have $K_2/L_2 > K_1/L_1$ such that $\kappa_2 > \kappa^*$ in accordance with Corollary 1. ■

A.7 Proof of Corollary 2

From Lemma 2, we have for $\kappa_t > 0$

$$\kappa_{t+1} > \kappa_t \quad \Leftrightarrow \quad G(\kappa_{t+1}) > G(\kappa_t). \tag{A.27}$$

Then, using (3.11) for $(t, t+1)$ and $(t-1, t)$ and (3.12) for B_t delivers $G(\kappa_{t+1}) > G(\kappa_t)$ if and only if (4.1) holds. ■

A.8 Proof of Proposition 5

1. Equation (5.6) is derived using essentially the same steps as in the derivation of (3.11) and taking (5.4), (5.1) and (5.3) into account. Uniqueness of the sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$ is guaranteed since $\tilde{G}(\kappa_t)$ inherits the properties of $G(\kappa_t)$ stated in Lemma 2.
2. Condition (3.15) delivers κ_ζ^* whereas (5.6) evaluated at $\kappa_t = \kappa_{t+1} = \kappa_\zeta^*$ gives B_ζ^*. Since the right-hand side of (5.6) decreases in $\hat{\tau}$ whereas the left-hand side increases, a larger $\hat{\tau}$ requires a larger B_ζ^*. ■

A.9 Proof of Proposition 6

1. The proof is the same as the one of Lemma 1 with $\tilde{\beta}$ replacing $\hat{\beta}$ and $\tilde{G}$ replacing G.
2. The proof is essentially the same as the proof of Claim 2 of Proposition 4. Here, the introduction of the pension scheme reduces savings of generation $t = 1$ such that $L_1 s_1 = K_2 < K_1 \left(1 - \delta + g^A(\kappa^*)\right)(1 + \lambda)$. Therefore $K_2/L_2 < K_1/L_1$ and, in accordance with Corollary 1, $\kappa_2 < \kappa_1 = \kappa^*$.
3. Claim 3 follows from the local stability properties proved as Claim 1 of this proposition. ■

A.10 Proof of Proposition 7

1. Since the steady-state efficient capital intensity does not depend on the growth rate of the labor force, I have $\kappa_\zeta^{*\prime} = \kappa_\zeta^*$. As to the effect on B_ζ^*, I consider a small change of λ such that

$$B_\zeta^{*\prime} \approx B_\zeta^* + \frac{\partial B_\zeta^*}{\partial \lambda}(\lambda' - \lambda) \tag{A.28}$$

is a valid approximation.

To derive $\partial B_\zeta^*/\partial\lambda$, consider (3.5) in conjunction with (5.4), (5.1), and (5.3). This gives

$$L_t \left[\frac{\beta}{1+\beta}\frac{1+\lambda}{\zeta+1+\lambda}\, w_t - \frac{1}{1+\beta}\frac{\zeta(1+\lambda)}{\zeta+1+\lambda}\frac{w_{t+1}}{R_{t+1}}\right] = K_{t+1}. \tag{A.29}$$

In a steady state, the latter may be written as

$$L_t\, MPS\left(\kappa_\zeta^*, \lambda, \zeta\right)\, w_t = K_{t+1}, \tag{A.30}$$

where the marginal propensity to save out of wage income, MPS, is

$$MPS\left(\kappa_\zeta^*, \lambda, \zeta\right) \equiv \frac{1+\lambda}{(1+\beta)(\zeta+1+\lambda)}\left[\beta - \frac{\zeta(1-\delta+g^A\left(\kappa_\zeta^*\right)}{R\left(\kappa_\zeta^*\right)}\right]. \tag{A.31}$$

Expressing (A.30) in terms of κ_ζ^* and B_ζ^* gives

$$\frac{MPS\left(\kappa_\zeta^*, \lambda, \zeta\right)}{1+\lambda}\,\tilde{w}(\kappa_\zeta^*) = \frac{G(\kappa_\zeta^*)}{B^*}. \tag{A.32}$$

Implicit differentiation delivers

$$\frac{dB_\zeta^*}{d\lambda} \quad = \quad \frac{B_\zeta^*}{1+\lambda}\left[1 - \frac{\partial MPS}{\partial \lambda}\frac{1+\lambda}{MPS}\right], \tag{A.33}$$

where the argument of MPS is $\left(\kappa_\zeta^*, \lambda, \zeta\right)$. Moreover, from (A.31), one readily verifies that

$$\frac{\partial MPS}{\partial \lambda}\frac{1+\lambda}{MPS} = \hat{\tau} \in (0,1). \tag{A.34}$$

Hence, $dB_\zeta^*/d\lambda > 0$. In response to a smaller λ, I have $B_\zeta^{*'} < B_\zeta^*$.

2. The decline in the growth rate of the labor force necessitates an adjustment of $\hat{\tau}$ from period $t = 2$ onwards. To study the effect on κ_2 consider (A.29) for $t = 1$ and $t = 2$, i. e.,

$$L_1\left[\frac{\beta}{1+\beta}\frac{1+\lambda}{\zeta+1+\lambda}\,w_1 - \frac{1}{1+\beta}\frac{\zeta(1+\lambda')}{\zeta+1+\lambda'}\frac{w_2}{R_2}\right] = K_2. \tag{A.35}$$

Since $\lambda' < \lambda$, the left-hand side of (A.35) is greater than in the steady state. Hence, $K_2/L_2 > K_1/L_1$. From Corollary1 it follows that $\kappa_2 > \kappa_\zeta^*$. Since $g^B\left(\kappa_2\right) < g^B\left(\kappa_\zeta^*\right) = \delta$, we also have $B_2 < B_\zeta^*$. ■

A.11 Proof of Proposition 8

1. Since the steady-state efficient capital intensity is independent of λ, we have indeed $\kappa_\zeta^* = \kappa_\tau^* = \kappa_\zeta^{*\prime} = \kappa_\tau^{*\prime}$. As to the effect on B_τ^*, I consider a small change of λ such that

$$B_\tau^{*\prime} \approx B_\tau^* + \frac{\partial B_\tau^*}{\partial \lambda}\left(\lambda' - \lambda\right) \tag{A.36}$$

is a valid approximation. To derive $\partial B_\tau^*/\partial\lambda$, consider (3.5) in conjunction with (6.1). Assuming a steady state, we have

$$L_t\, MPS\left(\kappa_\tau^*, \lambda, \tau\right)\, w_t = K_{t+1}, \tag{A.37}$$

where

$$MPS\left(\kappa_\tau^*, \lambda, \tau\right) \equiv \frac{1}{1+\beta}\left[\beta(1-\tau) - \frac{(1+\lambda)\tau\left(1-\delta+g^A\left(\kappa_\tau^*\right)\right)}{R\left(\kappa_\tau^*\right)}\right] \tag{A.38}$$

is the marginal propensity to save out of wage income. Expressing (A.37) in terms of κ_τ^* and B_τ^* gives

$$\frac{MPS\left(\kappa_\tau^*, \lambda, \tau\right)}{1+\lambda}\,\tilde{w}(\kappa_\tau^*) = \frac{G(\kappa_\tau^*)}{B_\tau^*}. \tag{A.39}$$

Implicit differentiation of the latter delivers

$$\frac{dB_\tau^*}{d\lambda} = \frac{B_\tau}{1+\lambda}\left[1 - \frac{\partial MPS}{\partial\lambda}\frac{1+\lambda}{MPS}\right], \tag{A.40}$$

where the argument of MPS is $(\kappa_\tau^*, \lambda, \tau)$. From (A.38), we have

$$\frac{\partial MPS}{\partial\lambda}\frac{1+\lambda}{MPS} = \frac{-1}{\frac{\beta(1-\tau)R(\kappa_\tau^*)}{(1+\lambda)\tau(1-\delta+g^A(\kappa_\tau^*))} - 1} < 0 \tag{A.41}$$

since $MPS\left(\kappa_\tau^*, \lambda, \tau\right) \in (0,1)$. Hence, $dB_\tau^*/d\lambda > 1$.

Comparing (A.40) to (A.33) at the steady state with $\tau = \hat{\tau}$ reveals that

$$\frac{dB_\zeta^*}{d\lambda} < \frac{B_\zeta^*}{1+\lambda} = \frac{B_\tau^*}{1+\lambda} < \frac{dB_\tau^*}{d\lambda}. \tag{A.42}$$

This completes the proof of Claim 1.

2. to be written

■

B The Phase Diagram

We develop the phase diagram in the (B_t, κ_t) - plane.

First, consider the locus $\Delta\kappa_t \equiv \kappa_{t+1} - \kappa_t$. From (3.11) it follows that

$$\Delta\kappa_t = 0 \equiv \{(B_t, \kappa_t)\,|\kappa_{t+1} - \kappa_t = 0\} \quad \Leftrightarrow \quad B_t = \frac{G(\kappa_t)}{\hat{\beta}\,\tilde{w}(\kappa_t)}. \tag{B.1}$$

By assumption, the locus $\Delta\kappa_t = 0$ is stable in the vicinity of the steady state, i. e., (A.15) holds. Then, $\hat{\beta}\,\tilde{w}'(\kappa^*) < G'(\kappa^*)/B^*$, and for all pairs (B_t, κ_t) satisfying (B.1) near (B^*, κ^*) we have $dB_t/d\kappa_t > 0$.

Next, consider the locus $\Delta B_t \equiv B_{t+1} - B_t$. From (A.18) we have for $B_t > 0$

$$\Delta B_t = 0 \equiv \{(B_t, \kappa_t)\,|B_{t+1} - B_t = 0\} \quad \Leftrightarrow \quad \delta = g^B(\kappa_{t+1}) = g^B(\phi^\kappa(\kappa_t, B_t)). \tag{B.2}$$

By (A.22), this locus is stable with monotonic convergence. Moreover, all pairs $(\kappa_t, B_t) > 0$ that satisfy (B.2) are implicitly given by

$$\kappa^* = \phi^\kappa(\kappa_t, B_t). \tag{B.3}$$

Implicit differentiation of (B.3) reveals that $dB_t/d\kappa_t = -\phi^\kappa_\kappa/\phi^\kappa_B < 0$ for all pairs $(\kappa_t, B_t) > 0$ that satisfy (B.3). The sign follows since $\phi^\kappa_\kappa > 0$ and $\phi^\kappa_B > 0$.

C Three Numerical Examples

These examples were computed with *mathematica*. All notebooks are available upon request.

I make the following three assumptions:

Assumption 1 *Per-period utility is logarithmic such that individual savings at t is given by (2.4).*

Assumption 2 *The production function of the final good F is CES, i. e.,*

$$F(Y_{K,t}, Y_{L,t}) = \Gamma\left[(1-\gamma)\,Y_{K,t}^{\frac{\varepsilon-1}{\varepsilon}} + \gamma Y_{L,t}^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{C.1}$$

where $\Gamma > 0$, $0 < \varepsilon < \infty$ is the constant elasticity of substitution between both inputs, and $0 < \gamma < 1$.

Assumption 3 *For $j = A, B$, the investment requirement function is given by*

$$i(q^j) = v_0\left(q^j\right)^2, \quad \textit{with } v_0 > 0. \tag{C.2}$$

I consider three environments parameterized by the elasticity of substitution: $\varepsilon = 1$, $\varepsilon = 1/2$, and $\varepsilon = 2$. When calibrating the model, I think of a period of 30 years. Therefore, I choose $\beta = .55$, a population growth rate $\lambda = .35$, and a depreciation rate of technological knowledge $\delta = .26$. These numbers correspond to per-period values of .98, .01, and .01, respectively. The depreciation rate of capital is set equal to unity. The distribution parameter is equal to $gamma = 2/3$ such that that share of output that accrues to the capital-intensive (labor-intensive) intermediate is 1/3 (2/3) if the elasticity of substitution is unity. The parameters of the input requirement function are $v0 = 1/2$ and $v = 2$. Finally, I set $\Gamma = 2.1$ with the implication, that the steady-state annual growth rates of per-capita magnitudes are 1.8% if for $\varepsilon = 1$. This growth rate is approximately consistent with the trend growth rates of most of today's industrialized countries.

Key Results For Three Examples.

Elasticity of Substitution	Steady State (κ^*, B^*)	steady-state growth rate $g^* = g^A(\kappa^*) - \delta$	average annual growth rate	eigenvalues (μ_1, μ_2)
$\varepsilon = 1$	$(3.6776, 14.2964)$	0.71869	1.8%	$(0.778386, 0.409777)$
$\varepsilon = 1/2$	$(1.81534, 11.181)$	0.79266	1.96%	$(0.746117, 0.335618)$
$\varepsilon = 2$	$(59.5637, 150.124)$	1.24951	2.7%	$(0.973239, 0.343434)$

Since both eigenvalues are strictly positive and smaller than unity, the steady state is a locally stable node. Moreover, observe that the growth rates increase in the elasticity of substitution, a result consistent with the Irmen (2009).

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