The First Order Approach to Moral Hazard Problems with Hidden Saving: The Case of CARA Utility

Sebastian Koehne*

University of Mannheim

Version: February 25, 2009

Abstract

For CARA preferences, this paper examines the first order approach to moral hazard problems in which the agent can secretly save and borrow. The paper shows that hidden saving constrains the concavity of the agent's problem even for CARA utility and additively separable effort disutility in an important way. We derive two sets of sufficient conditions for the validity of the first order approach in this setup. First, we strengthen the classic approach by Mirrlees (1979) and Rogerson (1985). We obtain a second set of conditions by using the theory of total positivity (Karlin 1968).

Keywords: moral hazard, hidden savings, first order approach, total positivity, log-convexity

JEL Classification: C61, D82, E21, H21

*Address: University of Mannheim, Department of Economics, 68131 Mannheim, Germany. Email: skoehne@rumms.uni-mannheim.de. I would like to thank my supervisors Ernst-Ludwig von Thadden and Nicola Pavoni for their encouragement and support. I am also grateful to Ian Jewitt, whose suggestions allowed an important improvement of this paper. I received further helpful comments from seminar participants at the University of Mannheim.

1 Introduction

The study of moral hazard problems is enormously simplified if one can rely on the first order approach. By replacing the incentive constraint with the necessary first order condition, this approach allows the application of Lagrangian methods. The seminal works of Rogerson (1985) and Jewitt (1988) have established sufficient conditions for the validity of this approach in the standard moral hazard problem.

However, for more general moral hazard problems very little is known. A particularly interesting class of problems for which this is true are moral hazard problems in which the agent can secretly save (and borrow). The conditions that validate the first order approach in the standard model unfortunately do not transfer to this framework (Kocherlakota 2004).[1] Obviously, the hidden asset decision makes the validity of the first order approach significantly more complex: In addition to making sure that the agent's utility is at a global maximum with respect to the effort decision, one has to show the same for the saving decision, and most importantly for joint deviations to different effort and saving levels. Typically, the agent would like to combine a reduction of effort with an increased savings level to insure against the worsened output distribution. Therefore, ruling out joint deviations is the main difficulty in proving that first order conditions are sufficient.

The present paper derives conditions that validate the first order approach for this class of problems when the agent has CARA utility. For the CARA case, the two-dimensional concavity requirement that validates the first order approach is equivalent to a strong concavity property in effort. More precisely, the agent's consumption utility in the second period will be jointly concave in effort and saving if and only if the negative consumption utility is log-convex in effort. Note that a univariate function is concave if the negative of the function is log-convex, but not vice versa. Therefore, establishing that utility is concave in effort (Rogerson 1985, Jewitt 1988) is necessary but not sufficient for the validity of the first order approach in the present problem.

We establish log-convexity of the agent's negative consumption utility and thus the validity of the first order approach in two ways. The first method uses a relatively strong assumption on the curvature of the output distribution function. This method shows that the first order

[1] Kocherlakota (2004) provides an example in which the first order approach to moral hazard with hidden saving fails even though the MLR and CDF conditions from Rogerson (1985) are satisfied. Kocherlakota studies logarithmic utility, but his argument applies to any utility function, including CARA utility in particular.

approach is valid for CARA utility if the distribution has monotone likelihood ratios (MLR) and a log-convex distribution function (LCDF). The LCDF assumption is stricter than Rogerson's (1985) CDF property and means that stochastic returns to effort are *strongly* decreasing.

The second method uses a result from the theory of total positivity (Karlin 1963, Karlin 1968). It shows that the first order approach is valid for CARA utility if the output distribution is totally positive of order 3, has monotone concave likelihood ratios and satisfies two other technical conditions. These additional conditions replace Jewitt's (1988) assumption that expected output be concave in effort. They are needed to amend Karlin's result on the preservation of concavity to the question of log-convexity. This method validates the first order approach in two steps. The first step establishes a strong concavity property for the wage scheme. Then, a finding from total positivity is used to transfer this condition to the desired result for the agent's choice problem. This establishes the first order approach for important examples that are not covered by the first method, the exponential distribution and the Poisson distribution, for instance.

Different from the setup with monetary effort costs studied in Fudenberg, Holmstrom, and Milgrom (1990), this paper assumes that preferences are additively separable into consumption utility and effort disutility. Monetary effort costs are technically much easier to study, but suffer from the result that the agent's effort choice will be independent of his wealth level. For most applications, this appears to be an unrealistic predicition. Of course, the assumption of constant absolute risk aversion might itself be highly debatable. Nonetheless, the CARA case marks an important step in understanding the validity of the first order approach in general. Moreover, CARA utility allows to connect the approach more easily to the well-known results by Rogerson (1985) and Jewitt (1988).

The current findings are related to the conditions derived in the pioneering paper by Abraham and Pavoni (2009). Abraham and Pavoni show that the first order approach is valid under hidden saving given that the agent's utility function features nonincreasing absolute risk aversion (NIARA) and the output technology satisfies the spanning condition from Grossman and Hart (1983) plus an additional condition concerning the elasticity of leisure. While their work is less restrictive with respect to the agent's preferences, it does not extend to distributions that violate the spanning condition. The present paper considers unrestricted output distributions instead. We identify the key role of log-convex distribution functions and show that Abraham and Pavoni's condition for the elasticity of leisure is a special example of this property. In addition, we provide a link to Jewitt's (1988) way of validating the first order approach. This allows us to establish the first order approach under hidden saving for important cases in which the LCDF condition does not hold.

The first order approach produces a very helpful characterization of optimal contracts. Questions on the monotonicity of consumption or the value of information can be answered immediately, and one finds many analogies to the model without hidden saving. One also finds important differences between the two models, as Abraham and Pavoni (2009) discuss in detail.

The first order approach is also important because it allows to give the multiperiod problem a tractable recursive structure, as discussed by Werning (2001, 2002), Kocherlakota (2004), and Abraham and Pavoni (2008), among others. Analytical results for the validity of this approach provide a theoretical foundation for this procedure.

The paper proceeds as follows: Section 2 describes the setup of the model. Section 3 contains the main results. Section 4 presents the proof and examples for the LCDF approach. Section 5 proves the total positivity approach and discusses two important examples. Finally, Section 6 concludes.

2 Setup

The relationship between Principal (P) and Agent (A) lasts for two periods. In the first period, A receives a fixed wage w_0 and buys s units of a (risk-free) one-period bond on an asset market which P cannot monitor. A positive value of s corresponds to saving, a negative value corresponds to borrowing. The interest rate is zero for simplicity. In the second period, A performs a hidden effort $e \in I$, where I is a real interval. This generates a stochastic output $x \in [\underline{x}, \overline{x}]$. The output is distributed according to the probability density $f(x, e)$, which is twice continuously differentiable and has full support for all $e \in I$. A receives the bond repayment s and an output-contingent wage $w(x)$.

A **contract** is a tuple $(w_0, w(\cdot), e, s)$ consisting of wages $(w_0, w(\cdot))$ and prescribed choices (e, s). w_0 is the (fixed) wage paid in period 1; $w(x)$ is the wage paid in period 2 given that output x has realized. The crucial assumption is that P observes neither the agent's bond holding s, nor his effort level e. Hence, the prescribed choices have to be incentive compatible. P offers a contract at the beginning of period 1. A's outside option delivers a utility of $\underline{U}$.

Preferences are as follows. P maximizes her expected profits, calculated as

$$-w_0 + \int_{\underline{x}}^{\overline{x}} (x - w(x)) f(x, e)\, dx. \tag{1}$$

A's preferences are additively separable into consumption utility u and effort disutility v, and are described by

$$u(w_0 - s) + \int_{\underline{x}}^{\overline{x}} u(w(x) + s) f(x, e)\, dx - v(e). \tag{2}$$

The agent's utility function has constant absolute risk aversion: $u(c) = -\exp(-\alpha c)/\alpha$, with $\alpha > 0$ constant. v is twice continuously differentiable and satisfies $v' > 0$, $v'' \geq 0$.

A contract is called **optimal** if it maximizes expected profits subject to incentive compatibility and individual rationality, i.e., if it solves the following problem:

$$\max_{w_0, w(\cdot), e, s} -w_0 + \int_{\underline{x}}^{\overline{x}} (x - w(x)) f(x, e)\, dx \tag{P1}$$

s.t.

$$(e, s) \in \operatorname*{argmax}_{(e', s') \in I \times \mathbb{R}} u(w_0 - s') + \int_{\underline{x}}^{\overline{x}} u(w(x) + s') f(x, e')\, dx - v(e') \tag{IC}$$

$$u(w_0 - s) + \int_{\underline{x}}^{\overline{x}} u(w(x) + s) f(x, e)\, dx - v(e) \geq \underline{U} \tag{PC}$$

If $(w_0, w(\cdot), e, s)$ is a solution to (P1), then so is $(w_0 - s, w(\cdot) + s, e, 0)$. Hence, we can assume $s = 0$ without loss of generality.

Problem (P1) is extremely intricate. The incentive constraint (IC) consists of a two-dimensional continuum of inequalities. It requires

$$\begin{aligned} &u(w_0) + \int_{\underline{x}}^{\overline{x}} u(w(x)) f(x, e)\, dx - v(e) \\ &\geq u(w_0 - s') + \int_{\underline{x}}^{\overline{x}} u(w(x) + s') f(x, e')\, dx - v(e') \end{aligned} \tag{3}$$

for all $e' \in I, s' \in \mathbb{R}$. To obtain a problem that can be solved by standard methods, one replaces the incentive constraint by the agent's first order necessary conditions. This gives rise to the following problem:

$$\max_{w_0, w(\cdot), e} -w_0 + \int_{\underline{x}}^{\overline{x}} (x - w(x)) f(x, e)\, dx \tag{P2}$$

s.t.

$$\int_{\underline{x}}^{\overline{x}} u(w(x)) f_e(x, e)\, dx - v'(e) = 0 \tag{FOCe}$$

$$u'(w_0) - \int_{\underline{x}}^{\overline{x}} u'(w(x)) f(x, e)\, dx = 0 \tag{FOCs}$$

$$u(w_0) + \int_{\underline{x}}^{\overline{x}} u(w(x)) f(x, e)\, dx - v(e) \geq \underline{U} \tag{PC}$$

Solutions to (P2) will be denoted by $(w_0^*, w^*(\cdot), e^*)$.

Replacing the true problem (P1) by the first order problem (P2) is a valid procedure only if their solutions coincide. Assuming that the solutions to (P1) are interior, this is the case if and only if the contracts solving (P2) are incentive compatible. A sufficient condition is that the agent's problem is concave at these contracts. The remainder of this paper will identify conditions under which this is the case.

3 Main results

Using λ, μ and ξ as the Lagrange multipliers associated with the constraints (PC), (FOCe), (FOCs), respectively, the first order condition of the Lagrangian of problem (P2) with respect to wages $w(x)$ is

$$0 = -f(x, e^*) + \mu u'(w^*(x)) f_e(x, e^*) - \xi u''(w^*(x)) f(x, e^*) + \lambda u'(w^*(x)) f(x, e^*), \quad x \in [\underline{x}, \overline{x}], \tag{4}$$

or equivalently

$$\frac{1}{u'(w^*(x))} = \lambda + \mu \frac{f_e(x, e^*)}{f(x, e^*)} + \xi \alpha, \quad x \in [\underline{x}, \overline{x}]. \tag{5}$$

The first two terms on the right-hand side of expression (5) are well-known from the standard moral hazard problem. The third term, $\xi\alpha$, is due to the agent's Euler equation, (FOCs).

The first order condition of the Lagrangian with respect to w_0 is

$$0 = -1 + \xi u''(w_0^*) + \lambda u'(w_0^*), \tag{6}$$

or equivalently

$$\frac{1}{u'(w_0^*)} = \lambda - \xi\alpha. \tag{7}$$

The multipliers λ, μ, ξ that satisfy the Kuhn-Tucker conditions (5), (7) are positive: $\lambda > 0$, $\mu > 0$, $\xi > 0$ (Abraham and Pavoni 2009).

Due to CARA utility, the structure of characterization (5) is very similar to the model without saving. This implies that concavity in effort follows from the same assumptions as in Rogerson (1985) and Jewitt (1988). However, the crucial question is *joint* concavity in effort and saving, which is a stronger property.

As noted above, the first order approach is valid if the agent's utility

$$(e, s) \mapsto u(w_0^* - s) + \int_{\underline{x}}^{\overline{x}} u(w^*(x) + s) f(x, e)\, dx - v(e) \tag{8}$$

is (jointly) concave at the contracts $(w_0^*, w^*(\cdot), e^*)$ that solve (P2). Since u is concave and v convex, it is sufficient to show concavity of the consumption utility in the second period,[2]

$$(e, s) \mapsto \int_{\underline{x}}^{\overline{x}} u(w^*(x) + s) f(x, e)\, dx. \tag{9}$$

Using $u(c) = -\exp(-\alpha c)/\alpha$, this function can be rewritten as

$$(e, s) \mapsto \exp(-\alpha s) \int_{\underline{x}}^{\overline{x}} u(w^*(x)) f(x, e)\, dx. \tag{10}$$

The multiplicative separability of (10) might hint that the concavity of the agent's problem does not differ from the model without saving. However, Proposition 2 shows that this is not the case. The result is a direct consequence of the following novel finding.

Lemma 1. *Let $G(e, s) = \exp(h(s))g(e)$. Assume g is twice continuously differentiable, negative*

[2] I give up the curvature generated by the effect of saving on first period utility and by the disutility of effort to make the problem more tractable. Ignoring the curvature of the disutility of effort involves no cost, because effort units can be normalized such that this function is linear.

and concave, and h is twice continuously differentiable and convex. Then the condition

$$g(e)g''(e) - (g'(e))^2 \geq 0 \quad \textit{for all } e \tag{11}$$

is sufficient for the concavity of G in (e, s). If h is linear, the condition is also necessary.

Proof. G is concave if and only if its Hessian has a nonpositive diagonal and a nonnegative determinant. Omitting all arguments, the Hessian equals

$$H = \begin{pmatrix} (h'' + h'h')\exp(h)g & h'\exp(h)g' \\ h'\exp(h)g' & \exp(h)g'' \end{pmatrix}. \tag{12}$$

The diagonal entries are nonpositive by assumption. The determinant is

$$\det H = (\exp(h)h')^2[gg'' - g'g'] + \exp(2h)h''gg'' \geq (\exp(h)h')^2[gg'' - g'g']. \tag{13}$$

Hence, $\det H$ is nonnegative if $gg'' - g'g' \geq 0$. If $h'' = 0$, then the converse is also true. □

The left-hand side of (11) is the second derivative of $\log(-g)$. Hence, Lemma 1 implies the following central insight.

Proposition 2. *For CARA utility, A's second period consumption utility $\int u(w(x)+s)f(x,e)\,dx$ is (jointly) concave in effort e and saving s if and only if $-\int u(w(x))f(x,e)\,dx$ is log-convex in effort e.*

If the negative of a univariate function is log-convex, then the function is concave, but not vice versa. Therefore, log-convexity of $-\int u(w(x))f(x,e)\,dx$ in effort is a stronger condition than concavity of $\int u(w(x))f(x,e)\,dx$ in effort. Hence, Proposition 2 establishes two results. First, it shows that the introduction of hidden savings constrains the concavity of the agent's decision problem even for CARA utility. Given the multiplicative separability of utility between effort and saving, this was not immediately obvious. Second, it shows that the analysis can be reduced to the effort dimension nevertheless. A strong one-dimensional concavity condition regarding effort implies the (joint) concavity of A's problem in effort and saving.

This paper identifies two sets of sufficient conditions for the desired log-convexity result. Theorem 1 uses monotonicity of the wage scheme and a relatively strong assumption on the curvature of the output distribution function. This procedure strengthens the classic approach by Mirrlees (1979) and Rogerson (1985) .

Assumption (MLR)**.** The likelihood ratio function $f_e(x,e)/f(x,e)$ is nondecreasing in x for all e.

The MLR condition is standard and simply means that more output is indicative of higher effort. This implies monotonicity of $w(\cdot)$ by (5).

Assumption (LCDF)**.** The distribution function $F(x,e)$ is log-convex in e for all x.

The LCDF condition states that $P(x \leq x'|e)$ is log-convex in e for all x'. In other words, $P(x > x'|e)$ is highly concave in e. Therefore, the LCDF property can be interpreted as strongly decreasing stochastic returns to effort.

The two conditions above imply the validity of the first order approach.

Theorem 1. *Let* $(w_0^*, w^*(\cdot), e^*)$ *be a solution to* (P2)*. Assume MLR, LCDF (and CARA). Then the agent's negative second period consumption utility,* $-\int u(w^*(x))f(x,e)\,dx$*, is log-convex in effort and the first order approach is valid.*

Proof. See Section 4. □

Note however that already the weaker assumption of convexity of the distribution function (CDF) has drawn some criticism. For instance, the condition does not hold for the very natural case where output is modeled as effort plus a random noise term (Jewitt 1988). Moreover, it fails for most common distributions, e.g., the normal distribution, Gamma distribution, Poisson distribution, and many others. In addition, CDF is typically violated if the link between effort and output shows relatively little noise (Conlon 2008).

Theorem 2 needs neither the LCDF assumption, nor the weaker CDF condition. It substitutes the convexity of the distribution function by a concavity property of the wage scheme. In spirit, this approach is therefore similar to Jewitt (1988).

Assumption (MCLR)**.** The likelihood ratio function $f_e(x,e)/f(x,e)$ is nondecreasing and concave in x for all e.

MCLR sharpens the MLR property. In addition to the assumption that information on effort grows with output, MCLR requires that the information grows at a decreasing rate.

The following two assumptions replace the LCDF condition.

Assumption (TP)**.** f is totally positive of order 3, i.e., for all $x_1 < \cdots < x_m$, $e_1 < \cdots < e_m$, with $1 \leq m \leq 3$, the determinant

$$\begin{vmatrix} f(x_1, e_1) & \cdots & f(x_1, e_m) \\ \vdots & \ddots & \vdots \\ f(x_m, e_1) & \cdots & f(x_m, e_m) \end{vmatrix} \tag{14}$$

is nonnegative.[3]

Even though somewhat opaque, total positivity is not too problematic. For instance, it holds for any exponential family of densities (under an appropriate parameterization). The use of total positivity and the following assumption will be become clear in the discussion of Theorem 2 below.

Assumption (TA, 'Technical assumption')**.** For all $e_1, e_2 \in I$, $e_1 < e_2$, there exists a $B \in \mathbb{R}$ such that

$$\frac{-\int u(w^*(x))f(x, e_2)\, dx}{-\int u(w^*(x))f(x, e_1)\, dx} = \frac{\int \exp(Bx)f(x, e_2)\, dx}{\int \exp(Bx)f(x, e_1)\, dx} \tag{TA1}$$

and

$$\int \exp(Bx)f(x, e)\, dx \quad \text{is log-convex in } e \in [e_1, e_2]. \tag{TA2}$$

Theorem 2. *Let* $(w_0^*, w^*(\cdot), e^*)$ *be a solution to* (P2)*. Assume MCLR, TP, TA (and CARA). Then the agent's negative second period consumption utility is log-convex in effort, hence the first order approach is valid.*

Proof. See Section 5. □

The proof consists of two main steps. The first step shows that the agent's negative utility function composed with the wage scheme, $-u(w^*(x))$, is log-convex in the output variable x. This property follows from the monotone concave likelihood ratio condition, MCLR, and the fact that u is a highly concave transformation of $1/u'$ for CARA utility.[4]

[3]For $m = 2$, the condition coincides with the MLR property. Therefore, total positivity of order 3 implies monotonicity of the likelihood ratios.

[4]$u(c) = \omega(1/u'(c))$, with $\omega(z) = (\alpha z)^{-1}$; $-\omega$ is log-convex.

The second step shows that log-convexity of the integrand $-u(w^*(x))$ generates log-convexity of the integral $-\int u(w^*(x))f(x,e)\,dx$ in the effort parameter e. This is done by a technique from the theory of total positivity (Karlin 1963, Karlin 1968). Karlin's result on the preservation of convexity has already been used and extended by Jewitt (1988). This paper applies Karlin's finding to the question of log-convexity. Assumption TA arises naturally in this context, because it implies that the family

$$e \mapsto A + \log\left(\int \exp(Bx)f(x,e)\,dx\right), \quad A, B \in \mathbb{R}, \tag{15}$$

contains enough convex functions such that any two points on the graph of the mapping

$$e \mapsto \log\left(-\int u(w^*(x))f(x,e)\,dx\right) \tag{16}$$

can be connected. (Compare this to the much more tractable family $e \mapsto A + B\int xf(x,e)\,dx$, $A, B \in \mathbb{R}$, which is relevant for the preservation of convexity.)

Condition (TA2) is straightforward to check for any given B, but verifying whether (TA1) has a solution is more involved. The following lemma provides a condition that is easier to handle.

Lemma 3. *Under MLR, equation* (TA1) *has a solution if*

$$\frac{-\int u(w^*(x))f(x,e_2)\,dx}{-\int u(w^*(x))f(x,e_1)\,dx} > \inf_{B\le 0} \frac{\int \exp(Bx)f(x,e_2)\,dx}{\int \exp(Bx)f(x,e_1)\,dx} \tag{TA1a}$$

Proof. See Section 5. □

Theorem 2 validates the first order approach for important examples that are not covered by Theorem 1. One such example is the exponential distribution (see Section 5). The biggest difficulty in applying Theorem 2 is the verification of condition (TA1a), which still involves the endogenous function $-\int u(w^*(x))f(x,e)\,dx$. However, verification of (TA1a) is structurally much simpler than testing for log-convexity directly.

4 Theorem 1: Proof and examples

Proof of Theorem 1. Partial integration yields

$$-\int_{\underline{x}}^{\overline{x}} u(w^*(x))f(x,e)\,dx = \int_{\underline{x}}^{\overline{x}} \frac{du(w^*(x))}{dx} F(x,e)\,dx + u(w^*(\overline{x})). \tag{17}$$

Equation (5) shows that w^* is nondecreasing under MLR. Hence, $du(w^*(x))/dx \geq 0$. By assumption, $F(x,e)$ is log-convex in e for all x. An application of the Cauchy-Schwarz inequality shows that the sum of two log-convex functions is log-convex (cp. Liu and Wang 2007). The same is true for the integral of log-convex functions. □

In applications, discrete output spaces $X = \{x_1, \ldots, x_n\}$, $x_i < x_j$ for $i < j$, are often studied. In this case, wages are vectors $(w_0, w_1, \ldots, w_n) \in \mathbb{R}^{n+1}$ and probability weights $(p_1(e), \ldots, p_n(e))$ replace the density function $f(x,e)$. All the previous results extend to the discrete setup without difficulty.

Example 1 (Rogerson 1985)**.** Rogerson's paper contains the following distribution function that satisfies MLR and CDF:

$$F_i(e) = F(x_i, e) = (x_i/x_n)^{e-\underline{e}}, \quad e \in I = (\underline{e}, \infty). \tag{18}$$

In fact, this distribution function is not only convex (CDF), but also log-convex (LCDF). Note

$$\log(F_i(e)) = (e - \underline{e}) \log\left(\frac{x_i}{x_n}\right), \tag{19}$$

which shows that $\log(F_i)$ is linear in e for all i.

Example 2 (Two outputs)**.** Consider the case with two possible outputs, $x_L < x_H$, and associated probabilities $p_L(e) = 1 - p(e)$, $p_H(e) = p(e)$ for some increasing function p, with $0 \leq p(e) \leq 1$. Since p is increasing, MLR is satisfied. LCDF is equivalent to the log-convexity of $1 - p(e)$. One example that satisfies this condition is the function $p(e) = 1 - \exp(-f(e))$, where $f : I \to (0, \infty)$ is increasing and concave.

Example 3 (Spanning condition)**.** Let $(\pi_{1h}, \ldots, \pi_{nh})$, $(\pi_{1l}, \ldots, \pi_{nl})$ be two probability distributions on $\{x_1, \ldots, x_n\}$ such that π_{ih}/π_{il} is nondecreasing in i. (This implies that π_h first order stochastically dominates π_l.) Let

$$p_i(e) = \Gamma(e)\pi_{ih} + (1 - \Gamma(e))\pi_{il} \tag{20}$$

for some increasing function Γ, with $0 \leq \Gamma(e) \leq 1$. Monotonicity of Γ, combined with the fact that π_{ih}/π_{il} is nondecreasing, yields MLR. Note

$$F_i(e) = F(x_i, e) = \sum_{j=1}^{i} p_j(e) = (1 - \Gamma(e)) \sum_{j=1}^{i} (\pi_{il} - \pi_{ih}) + \sum_{j=1}^{i} \pi_{ih}. \tag{21}$$

First order stochastic dominance implies $\sum_{j=1}^{i}(\pi_{il} - \pi_{ih}) \geq 0$. Therefore, LCDF holds if $1-\Gamma(e)$ is log-convex. Interestingly, this requirement is tantamount to

$$\frac{(\Gamma'(e))^2}{-\Gamma''(e)(1 - \Gamma(e))} \leq 1, \tag{22}$$

which is exactly the condition under which Abraham and Pavoni (2009) validate the first order approach for the spanning condition and NIARA utility. Abraham and Pavoni's reading of the property defined in (22) is that the Frisch elasticity of leisure must be less than one (Abraham and Pavoni 2009, p. 16). The present paper clarifies how condition (22) relates to Rogerson's CDF property and how it extends to cases without the restrictive spanning assumption (at least for CARA utility).

5 Theorem 2: Proof and examples

Theorem 2 is a consequence of the next two lemmas.

Lemma 4. *Assume MCLR (and CARA). Then for any solution to* (P2), $-u(w^*(x))$ *is log-convex.*

Proof. For CARA utility, we have

$$u(c) = -\frac{1}{\alpha} u'(c) = -\alpha^{-1} \left(\frac{1}{u'(c)} \right)^{-1}. \tag{23}$$

This implies

$$\log\left(-u(w^*(x))\right) = -\log(\alpha) - \log\left(\frac{1}{u'(w^*(x))} \right). \tag{24}$$

Equation (5) shows that $1/u'(w^*(x))$ is concave if $f_e(x,e^*)/f(x,e^*)$ is concave in x. Since the negative logarithm is decreasing and convex, this implies that the composition $-\log(1/u'(w^*(x)))$ is convex, which concludes the proof. $\square$

Lemma 5. [5] *Let $\phi : X \to \mathbb{R}_{++}$ be log-convex. Define $\Phi(e) = \int \phi(x) f(x,e)\,dx$. Suppose the following holds.*
(i) f is totally positive of order 3.
(ii) For all $e_1, e_2 \in I$, $e_1 < e_2$, there exist $A, B \in \mathbb{R}$ such that

$$\log\left(\int \exp(Bx) f(x,e)\,dx\right) \tag{25}$$

is convex in $e \in [e_1, e_2]$, and

$$\log(\Phi(e_i)) = A + \log\left(\int \exp(Bx) f(x,e_i)\,dx\right), \quad i = 1,2. \tag{26}$$

Then Φ is log-convex in I.

Proof. Let $e_1, e_2 \in I$, $e_1 < e_2$, and let A, B be real numbers that satisfy condition (ii).

Since ϕ is log-convex, the function $\log(\phi(x)) - (A + Bx)$ has at most two sign changes; if it has exactly two sign changes, the sign sequence is $+, -, +$. Due to the monotonicity of the exponential function, the same property holds for $\phi(x) - \exp(A)\exp(Bx)$. Since the number of sign changes is at most 2 and f is totally positive of order 3, the sign change property is inherited by the integral

$$\int [\phi(x) - \exp(A)\exp(Bx)]\, f(x,e)\,dx, \tag{27}$$

as Theorem 3.1 in Karlin (1968, p.21) shows. The integral can be rewritten as

$$\Phi(e) - \exp(A)\int \exp(Bx) f(x,e)\,dx. \tag{28}$$

[5] An earlier version of this paper included only a result for the exponential distribution. I am grateful to Ian Jewitt for pointing out this more general finding, which contains the exponential distribution as an important special case (see Example 5).

Applying the logarithm, the sign change property implies that $\log(\Phi(e))$ intersects the function

$$A + \log\left(\int \exp(Bx) f(x, e)\, dx\right) \tag{29}$$

at most twice; if it intersects twice, then the difference has the sign sequence $+, -, +$. By assumption (ii), the number of intersections is at least two. Hence, there are exactly two intersections, at e_1 and e_2, and $\log(\Phi(e))$ is smaller than $A + \log\left(\int \exp(Bx) f(x, e)\, dx\right)$ for all $e \in [e_1, e_2]$. Since the latter function is by assumption convex in $[e_1, e_2]$, $\log(\Phi(e))$ is smaller than the convex combination of $\log(\Phi(e_1))$ and $\log(\Phi(e_2))$ for all $e \in [e_1, e_2]$. □

Proof of Theorem 2. By Lemma 4, the function $-u(w^*(x))$ is log-convex. Assumptions TP and TA ensure that the conditions of Lemma 5 are satisfied for $\phi(x) = -u(w^*(x))$, $\Phi(e) = -\int u(w^*(x)) f(x, e)\, dx$. Therefore, $-\int u(w^*(x)) f(x, e)\, dx$ is log-convex. □

In addition to the above, the proof of Lemma 3 was also omitted in Chapter 3.

Proof of Lemma 3. Define $\Phi(e) = -\int u(w^*(x)) f(x, e)\, dx$. Since $\int \exp(Bx) f(x, e_i)\, dx$ is continuous in B, (TA1) can be solved if

$$\sup_{B\in\mathbb{R}} \frac{\int \exp(Bx) f(x, e_2)\, dx}{\int \exp(Bx) f(x, e_1)\, dx} > \frac{\Phi(e_2)}{\Phi(e_1)} > \inf_{B\in\mathbb{R}} \frac{\int \exp(Bx) f(x, e_2)\, dx}{\int \exp(Bx) f(x, e_1)\, dx}. \tag{30}$$

Since $-u(w^*(x))$ is decreasing and $e_1 < e_2$, MLR implies $\Phi(e_2) < \Phi(e_1)$. Hence, the first inequality in (30) follows from

$$\frac{\int \exp(0) f(x, e_2)\, dx}{\int \exp(0) f(x, e_1)\, dx} = 1. \tag{31}$$

It remains to show that positive values of B can be ignored when taking the infimum. Note that $\exp(Bx)$ is increasing in x for $B > 0$. Due to MLR, this implies

$$\frac{\int \exp(Bx) f(x, e_2)\, dx}{\int \exp(Bx) f(x, e_1)\, dx} \geq 1. \tag{32}$$

Hence, for a positive B the fraction is at least as large as for $B = 0$. □

The remainder of this section verifies the conditions of Theorem 2 for the Poisson distribution[6] and the exponential distribution.

[6] All results from Section 3 extend to discrete output spaces without problems.

Example 4 (Poisson distribution)**.** Let the output space be $X = \mathbb{N}_0$, with probability weights

$$p_k(e) = \frac{e^k \exp(-e)}{k!}. \tag{33}$$

Likelihood ratios $p'_k(e)/p_k(e)$ are increasing and linear in k, hence MCLR holds. Since the distribution is totally positive (Karlin 1968, p. 19), it remains to show TA. Condition (TA2), i.e., log-convexity of $\sum_{k=0}^{\infty} \exp(Bk) p_k(e)$, follows from

$$\sum_{k=0}^{\infty} \exp(Bk) p_k(e) = \exp(-e + e\exp(B)). \tag{34}$$

Now consider (TA1a) (which implies (TA1)). Let $e_1, e_2 \in I$, $e_1 < e_2$. Using (34), we obtain

$$\frac{\sum \exp(Bk) p_k(e_2)}{\sum \exp(Bk) p_k(e_1)} = \exp((e_2 - e_1)(\exp(B) - 1)). \tag{35}$$

The infimum of (35) over all nonpositive numbers B is $\exp(e_1 - e_2)$. Condition (TA1a) is thus satisfied if

$$\frac{-\sum u(w_k^*) p_k(e_2)}{-\sum u(w_k^*) p_k(e_1)} > \frac{\exp(e_1)}{\exp(e_2)}, \tag{36}$$

or equivalently

$$-\sum_{k=0}^{\infty} u(w_k^*) \frac{(e_2)^k}{k!} > -\sum_{k=0}^{\infty} u(w_k^*) \frac{(e_1)^k}{k!}. \tag{37}$$

This inequality is true, because $-u(w_k^*)$ is positive for all k and e^k is increasing in e. This implies (TA1a).

Example 5 (Exponential distribution)**.** Let f be the density of the exponential distribution with mean e, i.e.,

$$f(x, e) = \frac{1}{e} \exp\left(-\frac{x}{e}\right), \quad x \in (0, \infty),\ e \in I = (0, \infty). \tag{38}$$

Since f is totally positive (Karlin 1968, p. 19) and its likelihood ratios f_e/f are increasing and linear in x, it remains to verify TA. Note that we have

$$\int \exp(Bx) f(x, e)\, dx = \frac{1}{1 - eB} \tag{39}$$

for $eB < 1$. Therefore, this function is log-convex (for all $B \leq 0$), which establishes (TA2).

To simplify notation, write $\Phi(e) = -\int u(w^*(x))f(x,e)\,dx$. For the verification of (TA1), let $e_1, e_2 \in I$, $e_1 < e_2$. For $B \leq 0$, we have $e_1 B < 1$, $e_2 B < 1$. Hence, we can use equation (39) to see

$$\frac{\int \exp(Bx)f(x,e_2)\,dx}{\int \exp(Bx)f(x,e_2)\,dx} = \frac{1-e_1 B}{1-e_2 B}. \tag{40}$$

The infimum of (40) over all numbers $B \leq 0$ is e_1/e_2. Therefore, (TA1a) is satisfied if

$$\frac{\Phi(e_2)}{\Phi(e_1)} > \frac{e_1}{e_2}. \tag{41}$$

In other words, it remains to show that the function $e\Phi(e)$ is increasing in e.

Note that the likelihood ratio $f_e(x,e)/f(x,e)$ is linear in x. Using the definition of CARA utility and equation (5), this implies that solutions to (P2) take the form

$$u(w^*(x)) = -\frac{1}{c_1 + c_2 x}, \tag{42}$$

with positive constants c_1, c_2. Hence, $\Phi(e)$ equals

$$\Phi(e) = \int_0^\infty \frac{1}{(c_1 + c_2 x)e} \exp\left(-\frac{x}{e}\right)\,dx. \tag{43}$$

Using the substitution $x(t) = et - c_1/c_2$, one obtains

$$\Phi(e) = \frac{1}{ec_2}\exp(\frac{c_1}{ec_2})E_1(\frac{c_1}{ec_2}), \tag{44}$$

where E_1 denotes the E_1-function, $E_1(x) = \int_0^\infty e^{-t}/t\,dt$, $x \in (0,\infty)$. Thus, the function $e\Phi(e)$ equals

$$\frac{1}{c_2}\exp(\frac{c_1}{ec_2})E_1(\frac{c_1}{ec_2}) \tag{45}$$

and has the derivative

$$\frac{1}{ec_2} - \frac{c_1 \exp(\frac{c_1}{ec_2})E_1(\frac{c_1}{ec_2})}{e^2 c_2^2}. \tag{46}$$

Since $E_1(x) < \exp(-x)\log(1+\frac{1}{x})$, the derivative has the lower bound

$$\frac{1}{ec_2} - \frac{c_1}{e^2 c_2^2}\log\left(1 + \frac{ec_2}{c_1}\right). \tag{47}$$

This bound is nonnegative, since

$$\log\left(1+\frac{ec_2}{c_1}\right) \leq \frac{ec_2}{c_1}. \tag{48}$$

Therefore, $e\Phi(e)$ is increasing. This implies that condition (TA1a) is satisfied.

6 Concluding remarks

For the case of CARA utility, this paper has presented two sets of conditions that validate the first order approach for moral hazard problems with hidden saving.

The first method obtains conditions by building on log-convexity of the distribution function. This property can be interpreted as strongly decreasing stochastic returns to effort and is easy to check. However, it has the problem of failing for many standard distributions. The second method does not need log-convexity, but includes conditions that are more difficult to verify.

For future research, it seems worthwhile to explore alternative ways of validating the first order approach. It would also be very interesting to see whether the methods used in this paper can be extended to non-CARA preferences. Abraham and Pavoni (2009) can be interpreted in the sense that CARA can be relaxed under the spanning condition, but there are probably more general results.

References

Abraham, A., and N. Pavoni (2008): "Efficient Allocations with Moral Hazard and Hidden Borrowing and Lending: A Recursive Formulation," *Review of Economic Dynamics*, 11(4).

——— (2009): "Optimal Income Taxation and Hidden Borrowing and Lending: The First-Order Approach in Two Periods," University College London. Mimeo. January 2009. http://www.ucl.ac.uk/~uctpnpa/FOC.pdf.

Conlon, J. R. (2008): "Two New Conditions Supporting the First-Order Approach to Multi-Signal Principal-Agent Problems," *Econometrica*, forthcoming.

Fudenberg, D., B. Holmstrom, and P. Milgrom (1990): "Short-Term Contracts and Long-Term Agency Relationships," *Journal of Economic Theory*, 51(1), 1–31.

Grossman, S. J., and O. D. Hart (1983): "An Analysis of the Principal-Agent Problem," *Econometrica*, 51(1), 7–45.

Jewitt, I. (1988): "Justifying the First-Order Approach to Principal-Agent Problems," *Econometrica*, 56(5), 1177–1190.

Karlin, S. (1963): "Total Positivity and Convexity Preserving Transformations," in *Convexity, Proceedings of the Seventh Symposium in Pure Mathematics of the American Mathematical Society*, ed. by V. L. Klee. American Mathematical Society.

——— (1968): *Total Positivity. Volume I.* Stanford, California: Stanford University Press.

Kocherlakota, N. R. (2004): "Figuring out the impact of hidden savings on optimal unemployment insurance," *Review of Economic Dynamics*, 7(3), 541–554.

Liu, L. L., and Y. Wang (2007): "On the log-convexity of combinatorial sequences," *Advances in Applied Mathematics*, 39(4), 453 – 476.

Mirrlees, J. (1979): "The Implications of Moral Hazard for Optimal Insurance," Seminar given at Conference held in honor of Karl Borch, Bergen, Norway. Mimeo.

Rogerson, W. P. (1985): "The First-Order Approach to Principal-Agent Problems," *Econometrica*, 53(6), 1357–1367.

Werning, I. (2001): "Repeated Moral-Hazard with Unmonitored Wealth: A Recursive First-Order Approach," MIT. Mimeo. http://econ-www.mit.edu/files/1264.

——— (2002): "Optimal Unemployment Insurance with Unobservable Savings," MIT. Mimeo. http://econ-www.mit.edu/files/1267.