On the limits of the principle of maximum differentiation

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Abstract

This paper contributes to the literature on spatial demand models. Based on the seminal paper of Hotelling (1929) and its extension to quadratic demand by d'Asprement et al. (1979), I introduce elastic total demand by allowing for a reservation value of the consumers. When transportation costs are increasing and/or the valuation of the product is decreasing, firms have an incentive to leave the boundaries and move towards the quartiles of the location space.

1 Introduction

In his seminal paper, Hotelling (1929) presented a model with the contention that competition between two firms would lead them to minimize spatial differentiation. This result has been challenged by d'Aspremont, Gabszewicz and Thisse (1979) who showed that a minimum differentiation equilibrium fails to exist for sufficiently closely (but not identically) located firms. To amend this problem, they modify the original setup by changing the consumer's transportation costs from a linear to a quadratic function and find the opposite result: a principle of maximum differentiation.

Economides (1984) introduced a limited reservation price to Hotelling's original model and could thereby increase the location range for which price equilibria exist. His results also contain "relocation tendencies" that show the limits of the principle of minimum differentiation. Hinloopen/van Marrewijk (1999) add to this result by analyzing location equilibria in more detail.

This paper contributes to the literature by introducing a reservation price to a spatial demand model with quadratic transportation costs. As a result, the model balances two forces: the centralization tendency of a local monopolist and the differentiation tendency of full demand à la d'Aspremont et al. (1979). By moving away from each other, firms can decrease competition and thus increase the pie they share. This tendency will be valid as long as firms do not have to leave consumers unserved when moving away. With decreasing reservation price and/or an increasing transportation cost rate, firms find it optimal to locate more centrally. This tendency continues until firms reach the quartiles. Once market parameters are such that local monopolies are optimal, a range of optimal locations around the quartiles arises.

In chapter 2, I describe the setup of the model, chapter 3 analyzes all possible price equilibria while chapter 4 describes equilibria in location choice. The last chapter summarizes the results.

2 The Model

Consider a market with consumers uniformly distributed along the unit interval $[0, 1]$. There are two firms L and H that choose locations l and h, with $0 \leq l \leq h \leq 1$ and sell a product of reservation value v to the customers. Both firms face equal constant marginal cost $c < v$. Firm i's profit function is $\pi_i = (p_i - c)q_i$ where p_i is the price firm i charges and q_i her resulting aggregate demand, with $i\epsilon(l, h)$. Each consumer can choose between buying one unit of the good or not buying at all. When consuming the product, consumers face quadratic transportation costs between their own location x_c and the location of the firm they buy the good from. Transportation costs are scaled by the transportation cost rate $\frac{t}{2}$ (with $t \geq 0$). Net utility of consumers amounts to $U(p_l, l, x_c) = v - p_l - \frac{t}{2}(l - x_c)^2$ when buying from firm L and equals $U(p_h, h, x_c) = v - p_h - \frac{t}{2}(h - x_c)^2$ when buying from firm H.

The setup can be seen as a two stage game:(1) firms choose location, (2) firms compete in prices. An equilibrium in this dynamic game needs to be subgame perfect and can be identified by backward induction. I assume that consumers can observe both prices and locations of the firms. This implies that consumers choose to buy the good from firm i if $U(p_i, i, x_c) \geq U_c(p_j, j, x_c) \geq 0$, with $i, j\epsilon(l, h)$ and $i \neq j$. Hence, there is a potential demand for each firm, bounded by marginal consumers whose utility from consuming at this firm is equal to zero. The location of these *global marginal consumers* can be identified by solving $U(p_i, i, x_{mi}) = 0$ for x_{mi}, with $i\epsilon(l, h)$:

$$x_{mi}(p_i, i) = i \pm \sqrt{\frac{2}{t}(v - p_i)}. \tag{1}$$

Whenever the potential demand areas of both firms overlap, consumers will buy at the firm where the sum of price and transportation costs is smaller. In this case, firms are in a competitive world. Demand will then be limited by an additional marginal consumer. Call this the *within marginal consumer* and identify his location by solving $U(p_l, l, x_m) = U(p_h, h, x_m)$ for x_m:

$$x_m = \frac{(h + l)}{2} + \frac{p_h - p_l}{t(h - l)}. \tag{2}$$

Since transportation costs are convex, there is only one *within marginal consumer*[1]. As a consequence, consumers' net utility is monotone in distance

[1] See Economides (1984) for a general proof for a unique intersection of the value functions.

and firm L (H) will realize demand only to the left (right) of x_m. Also, if $x_m \leq 0$ ($x_m \geq 1$) firm L (H) will get zero demand. Since $v > c$, the firm facing zero demand can always improve by lowering her price. Hence, in equilibrium, both firms will get some positive fraction of her potential demand and the within marginal consumer will be on the interval $(0, 1)$. Demand is then limited by either a marginal consumer or the boundaries of consumer space, whatever comes first and can be decribed by:

$$\begin{aligned} q_l = &\quad \min\left\{l, \sqrt{2\tfrac{v-p_l}{t}}\right\} + \min\left\{\tfrac{h-l}{2} - \tfrac{p_l-p_h}{t(h-l)}, \sqrt{2\tfrac{v-p_l}{t}}\right\} \\ q_h = &\quad \min\left\{1-h, \sqrt{2\tfrac{v-p_h}{t}}\right\} + \min\left\{\tfrac{h-l}{2} + \tfrac{p_l-p_h}{t(h-l)}, \sqrt{2\tfrac{v-p_h}{t}}\right\} \end{aligned} \tag{3}$$

As a consequence of the minimum operator we have 4 different cases that will be analyzed separately in the following chapter. Crucial for the case distinctions is the market parameter $\theta \equiv \frac{v-c}{t}$. This parameter increases in the reservation value of the consumers (v) and decreases in marginal cost (c) and the transportation cost rate (t). It thus is a measure of the "intensity" of the competition in the market. Low θ implies a fragmented market with reduced competition, while an attractive and thus competitive market will be marked by a high θ.

Remark 1 *Throughout the paper I will speak about locations suggesting geographic distance. However, the results also hold when interpreting locations as product characteristics. In the latter case, the consumer's location pins down his preferred product type, and transportation costs describe the loss in utility when consuming a product not being identical with his preferred product type.*

3 Price Equilibria

There is no ex ante symmetry restriction on locations. When describing price equilibria, I also allow for asymmetric locations. However, when it comes to determining feasible ranges of these price equilibria, I assume that firms are identical in every decisive dimension. This means that if equilibrium profits are monotone in location, firms will choose identical levels of differentiation and I can focus on symmetric location equilibria. Price Equilibria will be classified as competitive, local monopolistic and touching, as proposed by Economides (1984).

3.1 Fully Competitive Equilibria

First let's look at equilibria where θ is so high and firms are sufficiently close that they fight for consumers between them. In other words, the within marginal consumer is binding. Depending on location and parameters, hinterlands are either fully covered (case 1) or not (case 2).

3.1.1 Full demand (case 1)

Assume θ to be so high that global marginal consumers are never binding. Figure 1 depicts this case: for two examplary location pairs, reservation price (v) and price plus transportation costs are ploted as functions of consumer location.

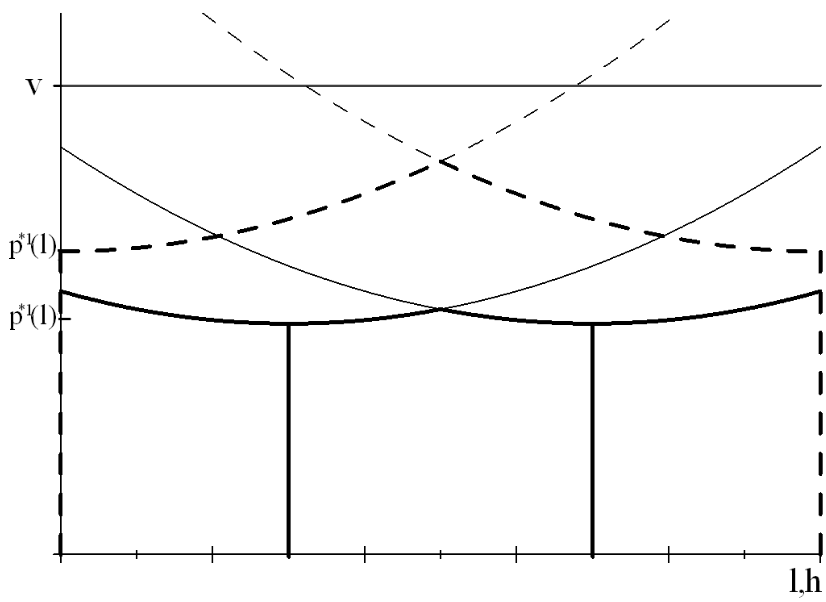

Figure 1: Price equilibria (solid,dashed: both case 1)

Even for the most extreme location, the within marginal consumer is binding and firms are in direct competition. This scenario is equal to the absence of a reservation price v and has been well-studied by d'Aspremont et al. (1979). The firms face demand according to

$$\begin{array}{r} q_l^1(l,h,p_l,p_h) = \frac{(h+l)}{2} + \frac{p_h - p_l}{t(h-l)} \\ q_h^1(l,h,p_l,p_h) = 1 - \frac{(h+l)}{2} - \frac{p_h - p_l}{t(h-l)} \end{array} \tag{4}$$

and realize the following profits

$$\begin{array}{r} \pi_l^1(l,h,p_l,p_h) = (p_l - c)\left(\frac{(h+l)}{2} + \frac{p_h - p_l}{t(h-l)}\right) \\ \pi_h^1(l,h,p_l,p_h) = (p_h - c)\left(1 - \frac{(h+l)}{2} - \frac{p_h - p_l}{t(h-l)}\right) \end{array} . \tag{5}$$

Combining first order conditions gives a unique Nash Equilibrium for given locations, where "^{*1}" denotes an equilibrium in the first case:

$$\begin{array}{r} p_l^{*1}(l,h) = c + \frac{t}{3}(h-l)\left(1 + \frac{h+l}{2}\right) \\ p_h^{*1}(l,h) = c + \frac{t}{3}(h-l)\left(2 - \frac{h+l}{2}\right) \end{array} . \tag{6}$$

However, if θ is sufficiently low, p^{*1} will not meet the following feasibility conditions of case 1:

$$\begin{array}{c} \sqrt{2\frac{v-p_l}{t}} \geq \max\left\{l, \frac{(h-l)}{2} - \frac{p_l-p_h}{t(h-l)}\right\} \\ \sqrt{2\frac{v-p_h}{t}} \geq \max\left\{1-h, \frac{(h-l)}{2} + \frac{p_l-p_h}{t(h-l)}\right\} \end{array} . \tag{7}$$

Lemma 2 *For*

$$l \geq l_{\min 1} = \left\{ \begin{array}{ccc} \frac{3}{2} - \sqrt{2\theta+1} & if & \theta \geq \frac{9}{32} \\ 1-\sqrt{2\theta} & if & \theta < \frac{9}{32} \end{array} \right. . \tag{8}$$

and

$$h \leq h_{\max 1} = \left\{ \begin{array}{ccc} \frac{1}{2} + \sqrt{2\theta+1} & if & \theta \geq \frac{9}{32} \\ \sqrt{2\theta} & if & \theta < \frac{9}{32} \end{array} \right. . \tag{9}$$

there is a Nash Equilibrium with firms choosing $p_l = p_l^{*1}$ and $p_h = p_h^{*1}$. Profits increase in spatial differentiation.

Proof. Plugging (6) into (7) and adding symmetry implies the marginal locations that still guarantee full market coverage when setting p^{*1}. Since $\frac{\partial \pi_l}{\partial l} < 0$ and $\frac{\partial \pi_h}{\partial h} > 0$ it is optimal to choose the most extreme location possible. ■

3.1.2 Uncovered fringes (case 2)

Next, I look at possible price equilibria where markets are so "rough" (θ is so low), that when competing at the center, firms find it optimal not to cover their own hinterland. Figure 2 depicts this case.

The demand function then looks as follows:

$$\begin{array}{c} q_l^2 = \frac{(h-l)}{2} + \frac{p_h-p_l}{t(h-l)} + \sqrt{2\frac{v-p_l}{t}} \\ q_h^2 = \frac{(h-l)}{2} - \frac{p_h-p_l}{t(h-l)} + \sqrt{2\frac{v-p_h}{t}} \end{array} . \tag{10}$$

Profits will be

$$\begin{array}{c} \pi_l^2 = (p_l - c)\left(\frac{(h-l)}{2} + \frac{p_h-p_l}{t(h-l)} + \sqrt{2\frac{v-p_l}{t}}\right) \\ \pi_h^2 = (p_h - c)\left(\frac{(h-l)}{2} - \frac{p_h-p_l}{t(h-l)} + \sqrt{2\frac{v-p_h}{t}}\right) \end{array} . \tag{11}$$

One can see that both profit functions only depend on the difference between locations, not on the absolute locations. Hence, individual locations can be reduced to distance $d \equiv h-l$. Taking first order conditions of (11) with respect to own price yields the following two equilibrium conditions:

$$\begin{array}{c} p_l = c + 2(p_h - c) - d^2\frac{t}{2} + \frac{(2v+c-3p_h)}{\sqrt{\frac{2}{t}(v-p_h)}}d \\ p_h = c + 2(p_l - c) - d^2\frac{t}{2} + \frac{(2v+c-3p_l)}{\sqrt{\frac{2}{t}(v-p_l)}}d \end{array} \tag{12}$$

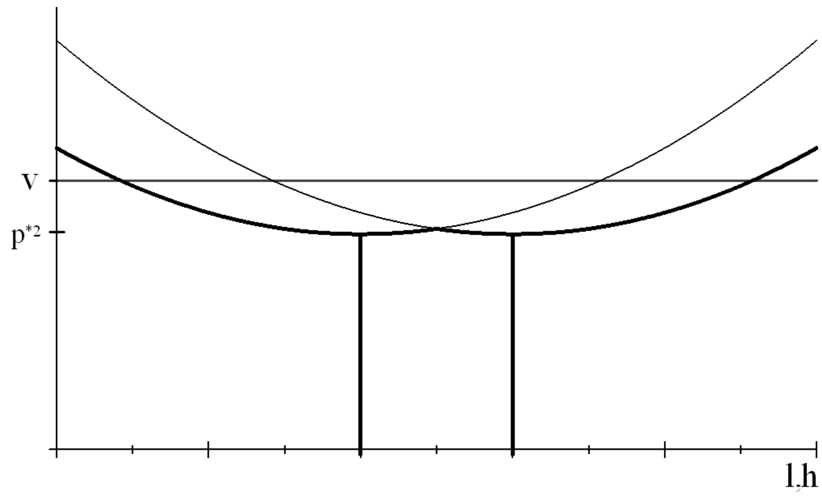

Figure 2: Price equilibrium case 2

These conditions yield

$$p^{*2} = \begin{cases} c + \frac{1}{3}t\left(\theta - 14d^2\right) - \frac{2}{3}t\sqrt{s}\cos\left(\frac{1}{3}\arccos\left(A\right)\right) & if \quad \sqrt{\frac{4}{3}\theta} \geq d \\ c + \frac{1}{3}t\left(\theta - 14d^2\right) - \frac{2}{3}t\sqrt{s}\cos\left(\frac{1}{3}\arccos\left(A\right) + \frac{1}{3}\pi\right) & if \quad \sqrt{\frac{4}{3}\theta} < d \end{cases}. \tag{13}$$

as symmetric mutual best response, where A and s are functions of d,t and θ (for details see Appendix). Feasibility of case 2 requires

$$\begin{array}{c} l > \sqrt{2\frac{v-p_l}{t}} > \frac{d}{2} - \frac{p_l - p_h}{td} \\ 1 - h > \sqrt{2\frac{v-p_h}{t}} > \frac{d}{2} + \frac{p_l - p_h}{td} \end{array}. \tag{14}$$

Lemma 3 *If*

$$\min\left\{\frac{l-2l^2}{2-2l} + l^2, -\frac{1}{2h}\left(-h^3 + 4h^2 - 4h + 1\right)\right\} > \theta \tag{15}$$

and

$$(h-l) < (h-l)_{\max} = \sqrt{\frac{24}{11}\theta} \tag{16}$$

there is a Nash Equilibrium with $p_l^* = p_h^* = p_l^{*2}$. *Profits increase in spatial differentiation.*

Proof. Combining (12) and (14) implies the symmetric marginal locations (for details see Appendix). We know $p_i^* = -q^2\left(p^*\right)\left(\frac{\partial q^2(p_i^*)}{\partial p}\right)^{-1}$ from $\frac{\partial \pi_i^2}{\partial p_i} = 0$. The envelope theorem implies $\frac{\partial p_i^*}{\partial l} = -\frac{\partial q^2(p^*)}{\partial l}\left(\frac{\partial q^2(p_i^*)}{\partial p}\right)^{-1} + q^2(p^*)\left(\frac{\partial q^2(p_i^*)}{\partial p}\right)^{-2}\left(\frac{\partial^2 q^2(p_i^*)}{\partial p \partial l}\right) < 0$. Using this result, also $\frac{\partial \pi_i^{*2}}{\partial l} = \frac{\partial p_i^*}{\partial l} q^2(p_i^*) + p_i^* \frac{\partial q_i^{*2}}{\partial l} < 0$. Hence, profits increase in spatial differentiation. ■

3.1.3 Corner solutions

When looking at feasibility conditions, it turns out that cases 1 and 2 can hardly occur on the same market. In addition, there are constellations where within marginal consumers are binding, but neither p^{*1} nor p^{*2} is feasible (see figure 3).

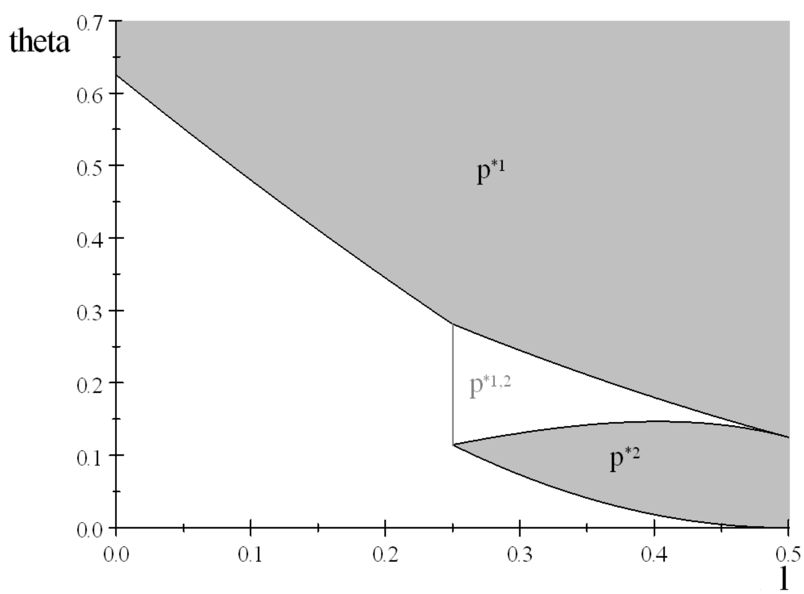

Figure 3: Feasible ranges for local monopolistic price equilibria

This is the case, when p^{*1} is too high to meet case condition of full demand, while p^{*2} is too low to meet case condition of uncovered fringes. This pushes the analysis towards corner solutions. Technically, a corner solution can only be included in case 1, since case 2 is defined as an open set. When adjusting both optimal prices to the best feasible price, they coincide and will just guarantee full demand [2].

[2] Observe, that best feasible response of subcase 2 also converges to this price.

Lemma 4 *If*

$$\frac{1}{4} < l < \max\left\{1-\sqrt{2\theta}, h-\sqrt{\frac{24}{11}\theta}\right\} \tag{17}$$

and

$$\frac{3}{4} > h > \max\left\{\sqrt{2\theta}, l+\sqrt{\frac{24}{11}\theta}\right\} \tag{18}$$

there is a Nash Equilibrium with

$$\begin{array}{c} p_l^* = p_l^{*1,2} = v - \frac{t}{2}l^2 \\ p_h^* = p_h^{*1,2} = v - \frac{t}{2}(1-h)^2 \end{array} . \tag{19}$$

Profits increase in spatial differentiation.

Proof. If $p^{*1,2}$ induces a Nash Equilibrium, both firms must have no incentive to increase or decrease the price. When decreasing price at $p^{*1,2}$, firms face q^1. Since $\frac{d\pi_l^1}{dp_l} \geq 0$ for all $p \leq p^{*1}$ and $p^{*1,2} \leq p^{*1}$ there is no incentive to decrease. When increasing price at $p^{*1,2}$, firms face q^2. Since $\frac{d\pi_l^2}{dp_l} \leq 0$ for all $p \geq p^{*2}$ and $p^{*1,2} \geq p^{*2}$ there is also no incentive to decrease. Feasibility conditions are implied by Lemma 2 and 3. ■

3.2 Local monopolies

Next we analyze the cases where θ is so low that firms can act as local monopolists. Demand can be asymmetric (case 3) or symmetric (case 4). See figure 4 for a graphical illustration.

3.2.1 Local monopolies with asymmetric demand (case 3)

If firms are too close to the ends to realize symmetric demand, they will face demand according to case 3:

$$\begin{array}{c} q_l^3 = l + \sqrt{\frac{2}{t}(v-p_l)} \\ q_h^3 = 1-h+\sqrt{\frac{2}{t}(v-p_h)} \end{array} . \tag{20}$$

Emerging profits are

$$\begin{array}{c} \pi_l^3 = (p_l - c)\left(l + \sqrt{\frac{2}{t}(v-p_l)}\right) \\ \pi_h^3 = (p_h - c)\left(1-h+\sqrt{\frac{2}{t}(v-p_h)}\right) \end{array} . \tag{21}$$

Optimal prices can be derived from the first order conditions immediately and are independent of the other firm's location.

$$\begin{array}{c} p_l^{*3} = \frac{1}{3}c + \frac{2}{3}v - \frac{1}{9}tl^2 + \frac{1}{9}l\sqrt{t^2l^2 + 6t(v-c)} \\ p_h^{*3} = \frac{1}{3}c + \frac{2}{3}v - \frac{1}{9}t(1-h)^2 + \frac{1}{9}l\sqrt{t^2(1-h)^2 + 6t(v-c)} \end{array} . \tag{22}$$

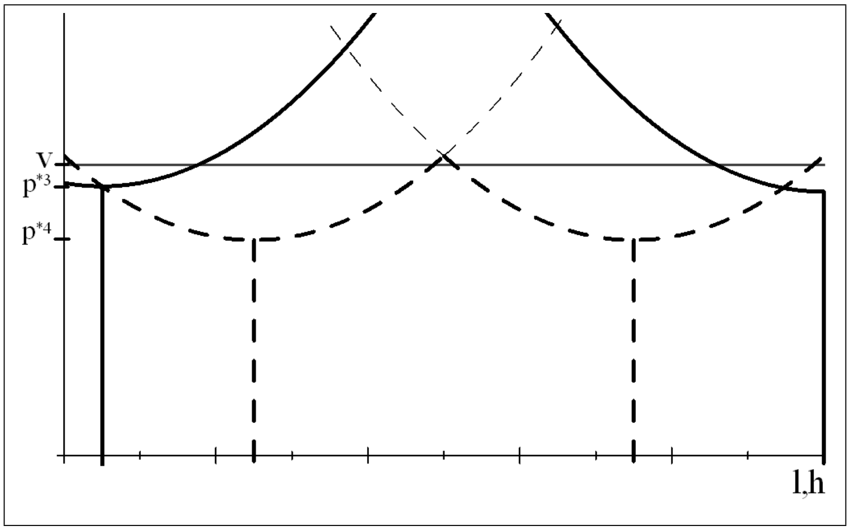

Figure 4: Price equilibria (solid: case 3, dashed:case 4)

Feasibility conditions of this case are:

$$\begin{array}{r} l < \sqrt{2\frac{v-p_l}{t}} < \frac{(h-l)}{2} - \frac{\Delta p}{t(h-l)} \\ 1-h < \sqrt{2\frac{v-p_h}{t}} < \frac{(h-l)}{2} + \frac{\Delta p}{t(h-l)} \end{array} . \tag{23}$$

Lemma 5 *If*

$$l < l_{\max 3} = \min\left\{\sqrt{\frac{2}{5}\theta}, 1 - \frac{1}{2}\sqrt{8\theta+1}\right\} \tag{24}$$

and

$$h > h_{\min 3} = \max\left\{1 - \sqrt{\frac{2}{5}\theta}, \frac{1}{2}\sqrt{8\theta+1}\right\} \tag{25}$$

there is a Nash-Equilibrium with both firms setting p^{*3}*. Profits are decreasing in spatial differentiation.*

Proof. Plugging (22) into (23) adding symmetry yields marginal locations (for details see appendix). By definition of this case, demands of the two firms are independent of each other. Hence, a Nash Equilibrium is a pair of individual best responses. It can be shown that for feasible locations $\frac{\partial \pi(p^{*3})}{\partial l} > 0$ and $\frac{\partial \pi(p^{*3})}{\partial h} < 0$. This is also intutitive, since moving away from the demand cutting boundary implies ceteris paribus increasing demand. ■

3.2.2 Local monopolies with symmetric demand (case 4)

Now let's look at the case where firms can realize the optimal monopolistic outcome. In this case demand is symmetric and amounts to:

$$q^4(p_i) = 2\sqrt{\frac{2}{t}(v - p_i)}. \tag{26}$$

The resulting profit function is

$$\pi^4(p_i) = (p_i - c)2\sqrt{\frac{2}{t}(v - p_i)}. \tag{27}$$

Optimal prices are

$$p_l^* = p_h^* = p^{*4} = \quad \tfrac{1}{3}c + \tfrac{2}{3}v \tag{28}$$

By construction, optimal price and profit are independent of location. Less intuitive, p^{*4} is even independent of t. Case conditions require

$$\begin{array}{r} \sqrt{2\frac{v-p_l}{t}} \leq \min\left\{l, \frac{(h-l)}{2} - \frac{\Delta p}{t(h-l)}\right\} \\ \sqrt{2\frac{v-p_h}{t}} \leq \min\left\{1-h, \frac{(h-l)}{2} + \frac{\Delta p}{t(h-l)}\right\} \end{array} . \tag{29}$$

Lemma 6 *For*

$$l \geq \sqrt{\frac{2}{3}\theta}. \tag{30}$$

and

$$l + \sqrt{\frac{8}{3}\theta} \leq h \leq 1 - \sqrt{\frac{2}{3}\theta}. \tag{31}$$

there is a Nash Equilibrium with both firms choosing p^{*4}. *Profits are independent of spatial differentiation.*

Proof. Marginal locations can be derived by plugging (28) into (29). By case assumptions both firms do not interfere in demand. Thus, a Nash Equilibrium is a pair of the individual best responses. $\frac{\partial \pi_l^4(p^{*4})}{\partial l} = \frac{\partial \pi_h^4(p^{*4})}{\partial h} = 0$. ■

3.2.3 Corner Solutions

Again, there is a gap between the two solution ranges as can be seen in figure 5. At θ-l-combinationss in the "gap" between the two regions, p^{*3} is too high to meet case condition of covering the fringes, while p^{*4} is too low to meet case condition of uncovered fringes. When adjusting both optimal prices to the best feasible price, they coincide and will just cover the fringes.

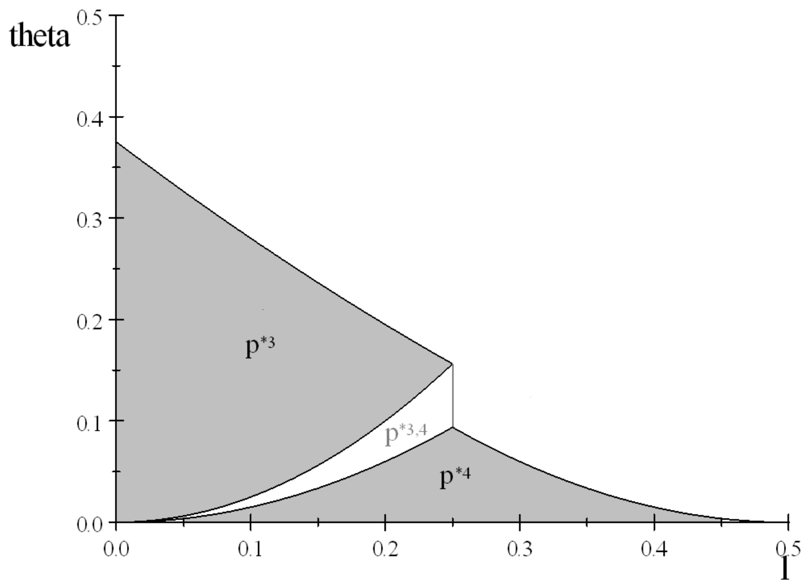

Figure 5: Feasible ranges for local monopolistic equilibria

Lemma 7 *If*

$$\sqrt{\frac{2}{3}\theta} > l \geq \sqrt{\frac{2}{5}\theta} \tag{32}$$

and

$$\sqrt{\frac{2}{3}\theta} > 1 - h \geq \sqrt{\frac{2}{5}\theta} \tag{33}$$

there is a Nash Equilibrium with

$$\begin{array}{c} p_l^* = p_l^{*3,4} = v - \frac{t}{2} l^2 \\ p_h^* = p_h^{*3,4} = v - \frac{t}{2} (1 - h)^2 \end{array} . \tag{34}$$

Profits are increasing in spatial differentiation.

Proof. If $p^{*3,4}$ induces a Nash Equilibrium, both firms must have no incentive to increase or decrease the price. When decreasing price at $p^{*3,4}$, firms face q^3. Since $p^{*1,2} \leq p^{*1}$ and $\frac{d\pi_l^3}{dp_l} \geq 0$ for all $p \leq p^{*3}$, price is already suboptimally low and hence there is no incentive to decrease. This suggests an incentive to increase the price. However, when increasing price at $p^{*3,4}$, firms face q^4. Since $\frac{d\pi_l^4}{dp_l} \leq 0$ for all $p \geq p^{*4}$ and $p^{*3,4} \geq p^{*4}$ there is also no incentive to decrease (see Appendix for details). ■

3.3 Touching Equilibria

Left to analyze are the regions, where neither a fully competitive nor a local monopolistic equilibrium exists. In a sense, those equilibria are also corner solutions. However, while so far corner solutions filled the transitions between covering and not covering the fringes, touching equilibria are at the borderline between full competition and local monopolies. This makes them a separate class of equilibria. Distinction is again whether hinterlands are fully covered or not.

3.3.1 Touching equilibrium with fringes covered

For locations in question, p^{*3} will be so low, that it contradicts case conditions of separated demand. On the other hand, p^{*1} is too high to cover the whole market. Corner solutions are only possible in the full demand world (case 1). Intuitively, optimal prices will be set such that demands of the two firms just touch.

Lemma 8 *For $l_{\min 1} > l > l_{\max 3}$ and $h_{\max 1} < h < h_{\min 3}$ there is a Nash equilibrium with both firms setting p^{*touch}. Profits decrease in spatial differentiation.*

$$\begin{array}{l} p_l^{*touch} = v - \frac{1}{2}t\left(\sqrt{\frac{2}{t}\left(v - p_h\right)} - (h-l)\right)^2 \\ p_h^{*touch} = v - \frac{1}{2}t\left(\sqrt{\frac{2}{t}\left(v - p_l\right)} - (h-l)\right)^2 \end{array} . \tag{35}$$

Proof. If p^{*touch} induces a Nash Equilibrium, both firms must have no incentive to increase or decrease the price. When decreasing the price, resulting demand will be q^1 and thus optimal price would be p^{*1}. Since profits are increasing in price for all $p < p^{*1}$ and infeasibility of p^{*1} implies $p^{*touch} < p^{*1}$, firms have no incentive to decrease the price. For p^{*touch} is already suboptimally low, there would rather be an incentice to increase the price. However, when increasing the price, marginal consumers "loose touch" and demand function is q^3. In this case, optimal price would be p^{*3} and profits are decreasing in price for $p > p^{*3}$. Since infeasibility of p^{*3} implies $p^{*touch} > p^{*3}$, firms have no incentive to increase the price. It can be shown that $\frac{\partial \pi_l^1\left(p^{*touch}\right)}{\partial l} > 0 \frac{\partial \pi_h^1\left(p^{*touch}\right)}{\partial h} < 0.$ for the feasible range of locations. ■

3.3.2 Touching equilibrium with fringes not fully covered

For locations in question, p^{*2} will be too high to cover the demand between the two firms as opposed to case assumption. Also p^{*4} will be too low to guarantee local monopolies. A corner solution can only be generated in case 2. Hence, p^{*2} will have to be lowered until inner global marginal consumers just touch. On the other hand, p^{*4} would have to be increased until these marginal consumers just loose touch. In the limit, both will coincide.

Lemma 9 *If*

$$\frac{24}{9}\sqrt{\theta} > h - l > \frac{24}{11}\sqrt{\theta} \tag{36}$$

and

$$l > \sqrt{2\frac{v - p_l}{t}}.and\ 1 - h < \sqrt{2\frac{v - p_h}{t}} \tag{37}$$

there is a (symmetric) Nash Equilibrium with both firms playing p^{*touch}. Profits increase in spatial differentiation.

Proof. In order to get a Nash Equilibrium there must be no incentive to deviate from the current price. Increasing the price firms face symmetric monopolistic demand q^4. Ideal price thus would be p^{*4}. Since $p^{*touch} > p^{*4}$ by definition of the "gap" and $\frac{\partial \pi^{*4}}{\partial p} < 0$ for all $p > p^{*4}$, there is no incentive to increase. When decreasing the price, firms face q^2 where p^{*2} would be the optimal price. Since $\frac{\partial \pi^{*4}}{\partial p} > 0$ for all $p < p^{*2}$ and infeasibility of p^{*2} implies $p^{*touch} < p^{*2}$, there is also no incentive to decrease. It can be shown that $\frac{\partial \pi_l^4 \left(p^{*touch}\right)}{\partial l} < 0 \frac{\partial \pi_h^4 \left(p^{*touch}\right)}{\partial h} > 0$ for the feasible range of locations. ■

4 Location Equilibria

Now let's move on to the first stage of the game where firms choose locations. According to the existence of different Nash Equilibria in the second stage we look at different levels of θ when solving for the first stage.

There is one range, where the analysis delivers a clear unique equilibrium.

Proposition 10 *For $\theta \geq \frac{5}{8}$, there is a unique Subgame Perfect Equilibrium with firms choosing to locate at $l^* = 0$ and $h^* = 1$, setting $p_l^* = p_l^{*1}(0,1) = p_h^* = p_h^{*1}(0,1) = c + \frac{t}{2}$.*

Proof. Using Lemma 1 it is optimal to choose the most extreme location possible, hence $l = l_{\min 1}$ and $h = h_{\max 1}$. Solving jointly for $l_{\min 1} = 0$ and $h_{\max 1} = 1$ implies $\theta \geq \frac{5}{8}$. ■

Once $\theta < \frac{5}{8}$, potentially multiple equilibria could occur. However, since in all equilibrium ranges profits are monotone in location and symmetric equilibria are existent for all locations, we can focus on symmetric equilibria.

Proposition 11 *For $\frac{20}{32} \geq \theta > \frac{9}{32}$ there are symmetric Subgame Perfect Equilibria with firms choosing $l = \frac{1}{2} - \sqrt{2\theta + 1}$ and $h = \frac{1}{2} + \sqrt{2\theta + 1}$ and $p = p_l^{*1} = p_l^{*touch}$. For $\frac{9}{32} \geq \theta > \frac{4}{32}$ there are symmetric Subgame Perfect Equilibria with firms choosing $l = 0.25$ and $h = 0.75$ and $p = p_l^{*touch}$. For $\theta \leq \frac{3}{32}$, firms will choose $l \in \left[\sqrt{\frac{2}{3}\theta}, \frac{1}{2} - \sqrt{\frac{2}{3}\theta}\right]$ and $h \in \left[\frac{1}{2} + \sqrt{\frac{2}{3}\theta}, 1 - \sqrt{\frac{2}{3}\theta}\right]$ and $p = p^{*4}$ $.\theta = \frac{3}{2}l^2$.*

All symmetric equilibrium locations can be identified in figure 5. Due to symmetry locations are only plotted for firm L.

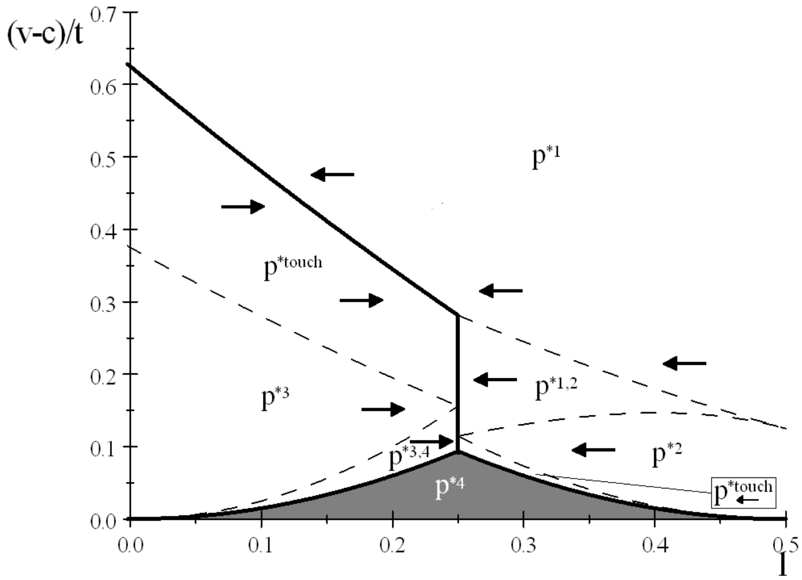

Figure 6: Symmetric Equilibrium locations

5 Conclusion

With high enough valuation, results are identical to the findings of d'Aspremont et al (1979). With decreasing valuation (or increasing transportation costs), the solutions of the full demand case start to become infeasible. As a result, it is no longer optimal to go for maximal differentiation. Rather, firms will move towards the most distant location where they can still realize full demand. This continues until both firms arrive at the quartiles (0.25, 0.75). If the valuation is even lower, firms first will prefer to stay at the quartiles and are then indifferent between the quartiles and a growing set of locations around them that allow for local monopolies.

To some extend the result is in line with the results from Rath/Zhao (2001) who also modelled elastic demand in a Hotelling set-up and found that optimal locations move towards the center when transportation costs increase. However, as opposed to their findings, in this paper there is a range of θ where firms stay at the middle of "their half", which is quite intuitive, since this is the best strategic position given their market half. It also means, that theta could be modified without distorting neither the market nor the market size. This could be the ground for political intervention that minimizes market distortions.

To some extend this result is in line with Rath/Zhao (2001) who analyze a model of quadratic transportation costs with a demand function that is inversely related to the price. They also find limitations to the principle of maximum differentiation. However, as opposed to my results, their setup does not allow for local monopolies and thus finds minimum differentiation for sufficiently low reservation values.

There are at least two interesting extensions of the model: One is introducing information asymmetries, the second one is introducing heterogenuous firms and thus asymmetric equilibria.

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6 Annex

6.1 Derivation of Lemma 2

Case conditions are:

$$\begin{array}{c} p_l(l,h) \leq v - \frac{1}{2}t\left(\sqrt{\frac{2}{t}\left(v-p_h\right)} - (h-l)\right)^2 \\ p_l < v - \frac{t}{2}l^2 \\ p_h < v - \frac{t}{2}\left(1-h\right)^2 \end{array} .$$

Plugging in optimal prices from (6) and rewriting gives in terms of θ these conditions become

$$\begin{array}{c} \frac{13}{72}h^2 - \frac{5}{36}hl + \frac{7}{18}h + \frac{13}{72}l^2 - \frac{11}{18}l + \frac{1}{18} \leq \theta \\ \frac{1}{3}(h-l)\left(1 + \frac{h+l}{2}\right) + \frac{1}{2}l^2 < \theta \\ \frac{1}{3}(h-l)\left(2 - \frac{h+l}{2}\right) + \frac{1}{2}(1-h)^2 < \theta \end{array}.$$

After rewriting in terms of l these conditions become

$$\begin{array}{c} l > \frac{5}{13}h - \frac{6}{13}\sqrt{2}\sqrt{-2h^2 - 2h + 13\theta + 6} + \frac{22}{13} \\ l > \frac{1}{2} - \frac{1}{2}\sqrt{12\theta - 2h(h+2) + 1} \\ h < \frac{1}{2} + \frac{1}{2}\sqrt{12\theta + 2l(4-l) + -5} \end{array}.$$

Applying symmetry, i. e. setting $h = 1 - l$, those conditions boil down to

$$\begin{array}{c} \frac{1}{2}l^2 - \frac{3}{2}l + \frac{5}{8} \leq \theta \\ \frac{1}{2}l^2 < \theta \end{array}.$$

Setting $1 - h = l = 0$ gives

$$\frac{5}{8} \leq \theta.$$

6.2 Derivation of Lemma 3

(A) Optimality

Taking first order conditions yields $(p_l - c) = \left(\sqrt{\frac{2}{t}(v - p_l)} + l - \frac{1}{2}\right) \frac{2t\left(\frac{1}{2}-l\right)\sqrt{\frac{2}{t}(v-p_l)}}{2\left(\frac{1}{2}-l\right)+\sqrt{\frac{2}{t}(v-p_l)}}$.

Combining optimality conditions and implying symmetric equilibria gives:

$p_l^3 + ap_l^2 + bp_l + c = 0$

with

$a = \left(14\left(\frac{d}{2}\right)^2 t - v - 2c\right)$

$b = \left(c^2 - 8c\left(\frac{d}{2}\right)^2 t + 4\left(\frac{d}{2}\right)^4 t^2 - 20\left(\frac{d}{2}\right)^2 tv + 2cv\right)$

$c = \left(2c^2\left(\frac{d}{2}\right)^2 t - c^2 v + 4c\left(\frac{d}{2}\right)^2 tv - 4\left(\frac{d}{2}\right)^4 t^2 v + 8\left(\frac{d}{2}\right)^2 tv^2\right)$

Using Cardano's formula (for a detailed discription see e. g. Delafuente (2000), p. 153):

Forming discriminant

$q = \frac{2}{27}a^3 - \frac{1}{3}ab + c = t^3\left(\frac{4984}{27}\left(\frac{d}{2}\right)^6 + \frac{424}{9}\left(\frac{d}{2}\right)^4\theta + \frac{40}{9}\left(\frac{d}{2}\right)^2\theta^2 - \frac{2}{27}\theta^3\right)$

$p = b - \frac{a^2}{3} = \frac{1}{3}t^2\left(-184\left(\frac{d}{2}\right)^4 - 32\left(\frac{d}{2}\right)^2\theta - \theta^2\right)$

$D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$

$= t^6\left(-\frac{80}{3}\left(\frac{d}{2}\right)^{12} - \frac{992}{9}\left(\frac{d}{2}\right)^{10}\theta + \frac{1360}{27}\left(\frac{d}{2}\right)^8\theta^2 + \frac{40}{9}\left(\frac{d}{2}\right)^6\theta^3 - \frac{16}{9}\left(\frac{d}{2}\right)^4\theta^4 - \frac{8}{27}\left(\frac{d}{2}\right)^2\theta^5\right)$

Since $t > 0$ the sign of dicriminant will only be positive when

$-\frac{5}{768}d^{12} - \frac{31}{288}d^{10}\theta + \frac{85}{432}d^8\theta^2 + \frac{5}{72}d^6\theta^3 - \frac{1}{9}d^4\theta^4 - \frac{2}{27}d^2\theta^5 > 0.$

Trying to find the nulls gives two complex and the following two real solutions:

$\theta_1 = 3\left(\frac{d}{2}\right)^2$

$\theta_2 = \sqrt[3]{54\left(\frac{d}{2}\right)^6} - 4\left(\frac{d}{2}\right)^2 < 0$

The second solution is infeasible since theta must be bigger than zero. Hence we are left with $\theta = 3\left(\frac{d}{2}\right)^2$. Checking derivative at this point reveals that this is a local maximum. For $\theta = 0$, the expression is still negative, hence it is negative for all $\theta > 0$. This means that the solution to the above problem is not possible with the standard Cardano approach, since this is the Casus irreducibilis, i. e. the case, when the transformation above does not deliver real solutions. Using Moivre's theorem the problem is solvable via an additional trigonometric transformation.

There are three possible solutions:

$z_1 = \sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}}\right)\right)$

$z_2 = -\sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}}\right) + \frac{1}{3}\pi\right)$

$z_3 = -\sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}}\right) - \frac{1}{3}\pi\right)$

with $p = z + \frac{1}{3}a$.

Plugging in and rewriting delivers

$p_1 = \frac{2}{3}t\sqrt{s}\cos\left(\frac{1}{3}\arccos(A)\right) - \frac{1}{3}t\left(14\left(\frac{d}{2}\right)^2 - \theta\right) + c$

$p_2 = -\frac{2}{3}t\sqrt{s}\cos\left(\frac{1}{3}\arccos(A) + \frac{1}{3}\pi\right) - \frac{1}{3}t\left(14\left(\frac{d}{2}\right)^2 - \theta\right) + c$

$p_3 = -\frac{2}{3}t\sqrt{s}\cos\left(\frac{1}{3}\arccos(A) - \frac{1}{3}\pi\right) - \frac{1}{3}t\left(14\left(\frac{d}{2}\right)^2 - \theta\right) + c$

with

$s = 184\left(\frac{d}{2}\right)^4 + 32\left(\frac{d}{2}\right)^2\theta + \theta^2$

and

$A = -\frac{\sqrt{2}\left(16\theta^3 - 240d^2\theta^2 - 636d^4\theta - 623d\right)}{8\left(23d^4 + 16d^2\theta + 2\theta^2\right)^{\frac{3}{2}}}$

The last solution (p_3) is not feasible, since it is negative for all applicable parameter values.

The first solutions (p_1) and (p_2) are feasible for some $\theta - d-$combinations:

->p_1 holds for $d^2 \geq \frac{4}{3}\theta$ and is increasing in its range.

->p_2 holds for $d^2 < \frac{4}{3}\theta$ and is increasing in its range.

(B) Feasibility

One approach would be plugging p^{*2} into feasibility constraint. Due to trigonometric expressions in p^{*2}, the results are quite messy.

Since we saw in part (A) that prices are monotone, we can use the following trick in order to get less messy results:

Minimum feasible price is $v - \frac{t}{2}l^2$. Hence it has to hold $v - \frac{t}{2}l^2 \leq p^{*2}$. At the margin it has to hold $v - \frac{t}{2}l^2 = p^{*2}$.

Hence, if the optimal price has only limited feasibility we can find marginal θ by plugging $p_l = v - \frac{t}{2}l^2$ into the optimalitiy condition

$(p_l - c) = \left(\sqrt{\frac{2}{t}(v - p_l)} + l - \frac{1}{2}\right) \frac{2t\left(\frac{1}{2}-l\right)\sqrt{\frac{2}{t}(v-p_l)}}{2\left(\frac{1}{2}-l\right)+\sqrt{\frac{2}{t}(v-p_l)}}$

and solve for θ.

$\theta = \frac{1}{2}\left(\frac{l-2l^2}{1-l} + l^2\right)$

Hence $\theta \leq \frac{1}{2}\left(\frac{l-2l^2}{1-l} + l^2\right)$

and $\theta \leq \frac{1}{2}\left(\frac{l-2(1-h)^2}{1-(1-h)} + (1-h)^2\right) = \frac{1}{2h}\left(h^3 - 4h^2 + 5h + l - 2\right)$

implies $v - \frac{t}{2}l^2 \leq p^{*2}$.

Maximum feasible price is $v - \frac{t}{8}(h-l)^2$. At the margin it has to hold $v - \frac{t}{8}(h-l)^2 = p^{*2}$.

Hence, we can find marginal θ by plugging $p_l = v - \frac{t}{2}\left(\frac{d}{2}\right)^2$ into the optimalitiy condition:

$\theta = \frac{11}{24}d^2$

rewriting in terms of l yields:

$\theta = \frac{11}{24}(h-l)^2$

Since, $\frac{\partial p_l}{\partial \theta} = \frac{2}{l^2} > 1$, $\theta \geq \frac{11}{24}(h-l)^2$ implies $v - \frac{t}{8}(h-l)^2 \geq p^{*2}$.

6.3 Derivation of Lemma 5

(A) Optimality

First order conditions of (21) imply:

$9p^2 + p\left(2tl^2 - 6(c+2v)\right) + (c+2v)^2 - 2tl^2v = 0$

under the condition that $p > \frac{1}{3}c + \frac{2}{3}v$. Due to quadratic equation, two solutions possible.

$p^{*3}(l) = \frac{1}{3}c + \frac{2}{3}v - \frac{1}{9}tl^2 \pm \frac{1}{9}l\sqrt{t^2l^2 + 6t(v-c)}$

Since $\frac{d^2\pi}{dp^2} < 0$ for all $c < p < v$ both solutions are local maxima.

However, $p > \frac{1}{3}c + \frac{2}{3}v$ only holds when using positive square root.

Hence, the only feasible solution is

$p^{*3}(l) = \frac{1}{3}c + \frac{2}{3}v - \frac{1}{9}tl^2 + \frac{1}{9}l\sqrt{t^2l^2 + 6t(v-c)}$.

(B) Feasibility

Pluggin in $p = p^{*3}$ into subcase condition $l \leq \sqrt{\frac{2}{t}(v - p_l)}$ and substituting $\theta = \frac{(v-c)}{t}$ gives:

$l \leq \sqrt{\frac{2}{3}\theta + \frac{2}{9}l^2 - \frac{2}{9}l\sqrt{l^2 + 6\theta}}$

$\Rightarrow 6\theta - 7l^2 \geq 2l\sqrt{l^2 + 6\theta}$

Since rhs is positive, there is no solution if $\theta < \frac{7}{6}l^2$.

Using this condition and then sqauring both sides yields

$\theta \geq \frac{5}{2}l^2$

Pluggin in $p = p^{*3}$ into subcase condition $\frac{1}{2} + \frac{p_l - p_h}{t(1-h-l)} - l > \sqrt{2\frac{v-p_l}{t}}$ and substituting $\theta = \frac{(v-c)}{t}$ gives

$\frac{2}{9}l\sqrt{l^2 + 6\theta} > -\left(\frac{1}{4} - l + \frac{7}{9}l^2 - \frac{2}{3}\theta\right)$

This holds for all p and l when $\theta \leq \frac{3}{8} - \frac{3}{2}l + \frac{7}{6}l^2$

If $\theta > \frac{3}{8} - \frac{3}{2}l + \frac{7}{6}l^2$ the condition becomes:

$36l^2\left(l - \frac{1}{2}\right)^2 > \left(6\theta - \left(\frac{9}{4} - 9l + 9l^2\right)\right)^2$

Solving quadratic equation taking into account possible negativities implies

$\frac{3}{8} - l + \frac{1}{2}l^2 > \theta$

Combining both subcase conditions yields

$\frac{3}{8} - l + \frac{1}{2}l^2 > \theta \geq \frac{5}{2}l^2$ or

$l \leq \min\left\{\sqrt{\frac{2}{5}\theta}l, 1 - \frac{1}{2}\sqrt{8\theta + 1}\right\}$