Optimal Fertility Decisions in a Life Cycle Model

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February 27, 2009

Abstract

This model is the first to solve for the optimal timing of childbirth and number of children in a continuous time framework simultaneously. The model depicts how changes in wage at different stages of an individual's life influence the timing decision of childbirth and the optimal number of children. When a woman wants to have more children, she decides to have them at a younger age. Medical research that extends the fecund life span induces women to have fewer children. A reduction of the parental leave due to daycare centers and child benefits increases the number of children. Women value labour more, when they face the risk of an unknown divorce. When the date of divorce is known, then labour supply after divorce does not jump. This paper also shows that divorce does not change the timing of childbirth directly, but it influences the number of children negatively and the number of children influences the timing.

1 Introduction

Three of the most significant socio-economic developments in virtually all the developed economies in the second half of the 20'th century were the large increases in female labour force participation, the falls in fertility rates and the increases in divorce rates. A number of exogenous factors clearly have played an important role in these, for example the growth in demand for female labour, the availability of the contraceptive pill, and changes in divorce laws that have made divorce easier and less costly to obtain. It seems also clear however that there are several possible interrelationships among these three developments: child care and work in the market are alternative uses of a mother's time and increasing wage rates raise the opportunity cost of children; the attempt to build a career could lead to postponing childbirth and having fewer children as a result of this; the perception of an increased chance that the marriage might end in divorce could lead to a decision to have fewer children. At the same time, there is considerable heterogeneity across households in respect of female market labour supply, even after controlling for wage rates and number and ages of children, and it does not seem adequate simply to regard this as due to preference heterogeneity.[1]

In this paper we develop a new theoretical framework to try to explore some of these interrelationships, and to consider possibe explanations for them, that are rooted in optimal intertemporal decision taking over the life cycle. A woman's human capital, and therefore her wage rate, is endogenous and depends first on the choice of how much formal education to acquire, and secondly on how much work experience to gain in the labour market. Both these decisions affect the timing and number of births, and in turn are affected by them because of the demands on time made by child care. We first set out a model which allows these interacting decisions to be formally analysed. We then extend it by analysing the effect on the timing and number of births of perceptions of the likelihood of divorce.

There is a large literature that asks how children affect such economic variables as demand patterns and consumption. In that context they examine intertemporal decisions and equality questions. For an overview of this literature see Browning (1992) and (Becker 1993). Most of the literature that deals with the effect of children on labour supply concentrates on female labour participation, because the effect on male labour market participation has so far been quite low.[2] Ward and Butz (1980) show empirically that couples time their births to avoid periods when the female's income is high. Heckman and Walker (1990) show that the negative (positive) relation between conceptions (fertility timing) and female wages is robust across a variety of empirical specifications, while they cannot prove that the same holds for male wages. Based on this literature we focus on the female as the utility maximizing individual throughout this paper.

In order to assess the costs of raising children, one has to take account of the timing of births. Labour market earnings depend on work experience. In an early study Happel et al. (1984) set up a model in which a woman works before she gives birth and gains labour market experience, and her income increases with experience. After giving birth a woman takes some time off to raise her child or children. When she re-enters the labour market, some of her experience has decayed by some constant factor. It is assumed to be zero for unskilled workers, in which case there is no timing preference. Otherwise a woman would want to either have children in the very beginning of her marriage, when she has not accumulated any labour experience before her marriage or shortly before her period of fecundity ends. In an empirical paper using Swedish data, Walker (1995) decomposes the total costs of children into the opportunity costs of not working, the foregone return for foregone human capital investment and the net direct. The model in this paper will take account of this decomposition and solve for the optimal timing in a continuous time framework.

Gustafsson (2001) gives a nice overview of the past theoretical and empirical research on the optimal timing of childbirth. Cigno (1991) analyses a dynamic

[1] See Apps and Rees (2009), chapters 1 and 5, where this is discussed at some length.

[2] Browning (1992) pp. 1449-1464

model in discrete time, in which the female's income depends on her education level as well as on labour market experience. He derives the optimality conditions that describe an optimal fertility profile, with the value of the number of children growing at the rate of interest. Along these lines he demonstrates that postponing conception raises the income loss and lowers the human capital loss of a birth, because income rises with labour experience. In order to go a step further in this paper we set up a model in continuous time, which allows us to find an explicit solution for the fertility timing and number of children. Blackburn et al. (1993) show theoretical linkages between a woman's fertility timing and her investments in human capital and income profile. A late childbearer accumulates more human capital when the discount rate is larger than the economy-wide growth rate of wages for late childbearers.

In our baseline model in the next section, we examine the effects of the income level on our two variables of interest: the timing of fertility and the number of children. We then go on to analyse how the return to labour market experience within the different life cycle phases affects the timing and number of births, which is new in this literature. We also have various cost parameters included for the purpose of deriving some policy implications.

The major part of the fertility literature is embedded in a deterministic framework. Exceptions are Newman (1983) and Hotz and Miller (1986). Drastic simplifications have to be made to keep these models manageable. As a consequence these models have bang-bang solutions, where the probability of giving birth is piled up either at the beginning of marriage or at the end of a woman's period of fecundity. Our model introduces some stochastic elements by introducing the possibility of divorce. We then show how this possibility influences the optimal timing and number of childbirths, and this appears to be new to the literature.

2 The Baseline Model

We assume that the working life of a representative woman falls into 3 stages (Figure: 1):

1. During the first phase $t \in [t_1; t_2]$ she works full-time. A utility function that accounts for leisure and consumption that can solve for the optimal control problem is quasilinear in leisure and linear in consumption $x_i(t)$. Total time is assumed to be Ψ, and labour is denoted by $l_i(t)$, where i is the subscript for the present phase the representative is in. An individual gains utility from consuming the representative good and leisure: $u_1 [x_1(t), l_1(t)] = x_1(t) + \ln [\Psi - l_1(t)]$. The price of consumption is normalized to 1. All income is consumed, hence the budget constraint is given by $w(\theta, L(t))l_1(t) = x_1(t)$, where the income $w(\theta, L(t))$ depends on ability

θ and labour experience gained thus far, $L(t) = \int_{t_1}^{t} l(t)dt$. Labour experience $L(t)$ is the state variable of this problem and to simplify notations it is denoted $L(t) = L_t$. L_0 is assumed to be zero, hence the the first income $w(\theta, 0)$ depends soley on ability. Education could also be part of this ability parameter. The appendix shows how a proceeding education phase influences fertility; the timing when she enters the labour market and her initial become endogenous. It reflects how, how flexible this model set-up is, and that it can be used for a wide variety of policy evaluations that affect fertility. In order to keep the model manageable to avoid adding more phases, we make the simplifying assumption that all children are born at the same time t_2 and do not require any child-care after t_3. The length of phase 3 has length $h(k)$ and depends on the number of children k. The decisions, how when to have children and how many children one wants to have depend on each other in real life. This is also reflected by this model setup as that t_2 and k are derived simultaneously.

2. During phase two, when $t \in [t_2, t_3]$, the woman has children and works part-time. When she is married and does not get divorced, which we assume in the baseline model, then time costs for k children that have been born at t_2 are $c(k, t_2)$ and the monetary costs are $m(k, t_2)$. After divorce a woman's time costs and monetary costs increase to $c^d(k, t_2)$ and $m^d(k, t_2)$, respectively. Having k children introduces not just costs but also benefits from having children during phase two and three $v_i(k)$; $i \in (2, 3)$. The utility function is given by $u_2 [x_2(t), l_2(t)] = x_2(t) + \ln [\Psi - l_2(t) - c(k, t_2)] + v_2(k)$. The labour income is consumed partly by the mother and partly by her children, the budget constraint is therefore $w(L_t)l_2(t) = x_2(t) + m(k, t_2)$, where the monetary costs for the mother are smaller than the total monetary costs of having k children, because the husband is assumed to contribute his part as well. How much he contributes depends on different aspects such as his own income, outside options for having k children with this particular woman, and the intra-household distribution. This model could be extended to take these complex issues into consideration. They are left open for further research.

3. During the last phase $t \in [t_3, T]$, the individual works full-time again. After t_3 children are older and do not have to be looked after. A woman consumes the consumption good $x_3(t)$, leisure $[\Psi - l_3(t)]$ and retrieves utility from having k children $v_3(k)$, thus $u_3 [x_3(t), l_3(t)] = x_3(t) + \ln [\Psi - l_3] + v_3(k)$. The budget constraint in this phase is $w(L_{t_3})l_3(t) = x_3(t)$. The wage depends on the labour experience accumulated until the end of phase 2. We assume that the wage is constant during this phase for simplicity. We also solved the model for a non-constant wage, but the main results do not change. Empirically one can observe that wage often even decreases before retirement, hence labour experience gained then does not pay off. At time T the planning horizon ends. The retirement shall not play any role in this analysis.

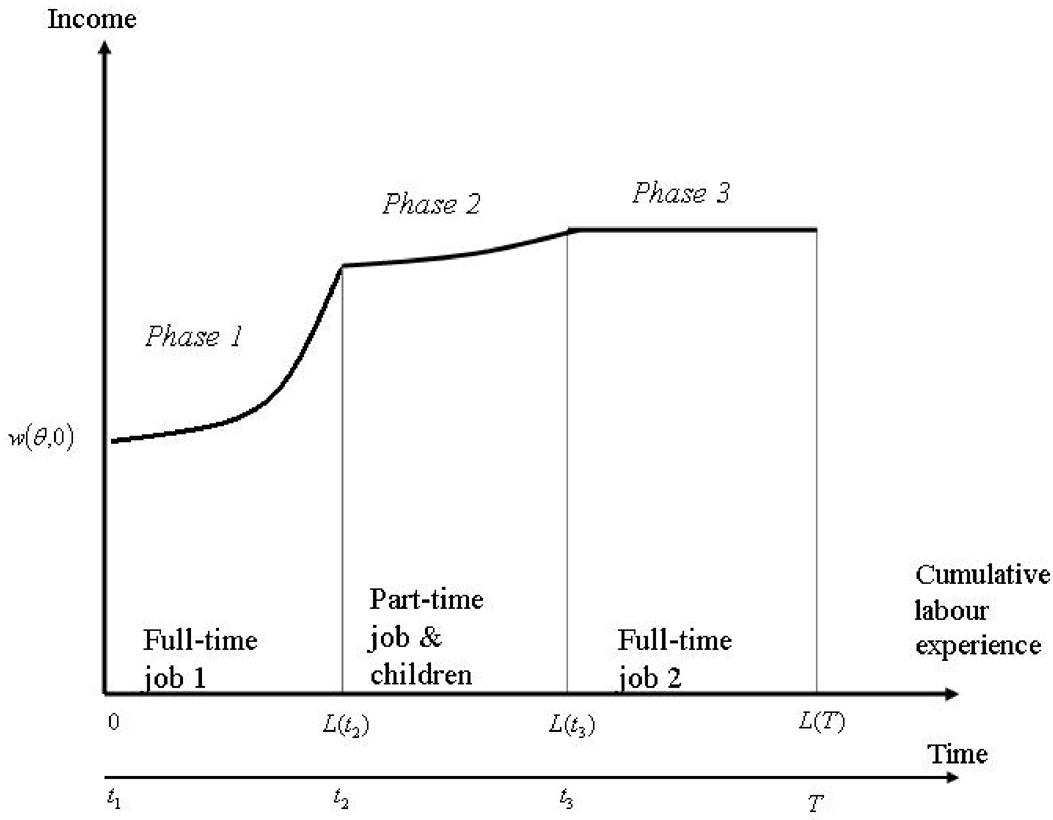

Figure 1: The per period income over a life-cycle.

The Hamiltonian for phase $i \in [1, 2, 3]$ is given by $H[x_i(t), l_i(t), \eta_i(t)] = u_i + \eta_i(t)l_i(t)$, where $\eta_i(t)$ is the co-state function of this optimal control problem. During the last phase $\eta_3(t) = 0$, because the wage rate is constant then. The derivative of the income with respect to labour experience is denoted as $\frac{\partial w_i(L_t)}{\partial L_t} = \alpha_i\,(, L_t)$. $\alpha_i\,(L_t)$ is larger during phase 1 than during phase 2 when a mother works part-time. A possible income scheme is shown by figure 1.

The planning horizon begins at $t = t_1$ and ends at $t = T$; both exogenous. t_2 is determined in the baseline model, t_3 shall be equal to t_2 plus $h(k)$, which is time independent and depends on the number of k children; $t_3 = t_2 + h(k)$. $h(k)$ characterizes the length of time of parental leave. For simplicity however, and because we are not interested in the choice of interval between biths, we assume that all children are born at t_2. We do not assume that skills deteriorate during phase two as Happel et al. (1984), but that could be another possible extension.

We solve the problem for each of the three phases of a woman's life backward from the last. We develop necessary conditions for this problem. First we take $t_2 \in [t_1, T]$ and $k > 0$ as given and solve for the optimal consumption and labour supply. In a next step we characterize the optimal time of childbirth t_2. By a theorem of Hestens, given the problem with t_2 and k fixed, we can define $\eta_i(t)$ on $[t_i, t_{i+1}]$; $i = 1, 2$ as the co-state variables of labour experience.[3]

2.1 Solving the model

Phase 3: $t \in [t_3, T]$

An individual's objective is to maximize $\int_{t_3}^{T} \{x_3(t) + \ln[\Psi - l_3] + v_3(k)\} \, dt$ subject to the budget constraint. The Lagrangian is

$$\Gamma[x_3(t), l_3(t)] = x_3(t) + \ln[\Psi - l_3] + v_3(k) + \lambda_3(t)[w(L_{t_3})l_3(t) - x_3(t)] \tag{1}$$

where $\lambda_3(t)$ is the Lagrangian multiplier for phase three. For simplicity we assume no discounting. A positive discount rate complicates the analysis unnecessarily and leads to a decrease in labour supply, because experience is valued less. A proof follows the same lines as in the appendix of Scholz (2008). The constant labour supply and consumption can be expressed in terms of the wage rate achieved at t_3.

$$l_3^* = \Psi - \frac{1}{w(L_{t_3})} \tag{2}$$

$$x_3^* = \Psi w(L_{t_3}) - 1 \tag{3}$$

Phase 2: $t \in [t_2, t_3]$

The computations are more refined in this section as that labour experience obtained within this phase has a future return. The objective here is to maximize $\int_{t_2}^{t_3} \{x_2(t) + \ln[\Psi - l_2(t) - c(k, t_2)] + v_2(k)\} \, dt + V_3^*$ subject to the budget constraint $w(L_t)l_2(t) = x_2(t) + m(k, t_2)$ and $\dot{L}_t = l_2(t)$. V_3^* is the optimally chosen utility stream from t_3 to T, given some labour experience level L_{t_2}. The choice of labour in this phase determines L_{t_3} and thus effects V_3^*. The Lagrangian is

$$\begin{aligned} \Gamma_2[x_2(t), l_2(t)] \;=\;& x_2(t) + \ln[\Psi - l_2(t) - c(k, t_2)] + v_2(k) \\ & + \eta_2(t)l_2(t) + \lambda_2(t)[w(L_t)l_2(t) - x_2(t) - m(k, t_2)] \end{aligned} \tag{4}$$

From the first order condition of labour and the general optimal control condition, where the time derivative of the co-state is equal to the negative

[3] see Takayama p.658

Hamiltonian's derivative with respect to the state variable (labour experience), we determine the following two expressions after substituting the optimality condition for consumption $\lambda_2(t) = 1$. Time derivatives are denoted by a dot above a time dependent function.

$$l_2(t) = \Psi - c(k, t_2) - \frac{1}{\eta_2(t) + w(L_t)} \tag{5}$$

$$\dot{\eta_2}(t) = -\dot{w}(L_t) \tag{6}$$

The transversality condition here is an expression of the co-state at t_3. Working an additional hour at t_3 increases her income and has a future return of

$$\eta_2(t_3) = \frac{\partial w(L_t)}{\partial L_t}|_{t=t_3} \int_{t_3}^{T} l_3 dt \tag{7}$$

Given the tranversality condition (7) and the transformation

$$l_2(t) \frac{\partial w(\theta, L_t)}{\partial L_t} = \dot{w}(L_t) \tag{8}$$

we can transform (6) in a way such that the co-state function becomes

$$\eta_2(t) = w(L_{t_3}) - w(L_t) + \alpha_2(t_3) \left[L_T - L_{t_3}\right] \tag{9}$$

where $\alpha_2(t_3) = \frac{\partial w(L_t)}{\partial L_t}|_{t=t_3}$. Using (5) and (9) one can solve for the optimal labour supply, which is time independent and its consumption counterpart, which does depend on time,

$$l_2^* = \Psi - c(k, t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3) \left[L_T - L_{t_3}\right]} \tag{10}$$

$$x_2^*(t) = w(L_t)[\Psi - c(k, t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3) \left[L_T - L_{t_3}\right]}] - m(k, t_2) \tag{11}$$

The labour supply is also independent from time in phase 1, which we show next. This result is driven by a decreasing return of experience, as the length of time between any t and T, when earlier accumulated experience pays off, decreases. On the other side income increases with experience, which would increase labour supply. Both effects are equally strong and cancel out. This result can be compared to the pricing of a monopolist that produces a single good and learns through production, which is reflected by decreasing unit costs. At each period it sets an optimal price such that its marginal revenue equals the marginal costs at the end of its planning horizon. Given a constant elasticity of demand its price is constant, even though its marginal costs decrease. (Spence 1981)

This feature is useful considering the fact that we do not view changes in labour supply from period to period in reality either. Hence this model is more realistic owing to a derivable constant labour supply. Furthermore we derive an increasing consumption function mimicing reality.

Phase 1: $t \in [t_1, t_2]$

An individual's objective is to maximize $V_1 = \int_{t_1}^{t_2} \{x_1(t) + \ln[\Psi - l_1(t)]\} \, dt + V_2^* dt$ subject to the budget constraint and $\dot{L}_t = l_1(t)$. The choice of labour in this phase determines L_{t_2} and influences the utility stream after $t = t_2$, which is denoted by V_2^*. The solution to the problem is

$$l_1^* = \Psi - \frac{1}{w(L_{t_2}) + \alpha_1(t_2)\,[L_T - L_{t_2}]} \tag{12}$$

$$x_1^*(t) = w(L_t)\left[\Psi - \frac{1}{w(L_{t_2}) + \alpha_1(t_2)\,[L_T - L_{t_2}]}\right] \tag{13}$$

The co-state functional for phase 2 has been derived following the same lines that have led to (7)

$$\eta_1(t) = w(L_{t_2}) - w(L_t) + \alpha_1(t_2)\,[L_T - L_{t_2}] \tag{14}$$

Conclusively we are able to determine the labour supplies for each phase and thus expressions for cumulative labour supplies at the end of phases 1-3. These expressions are needed, when solving for the timing of fertility. They are given by the integrals of instantaneous labour supplies (2), (10) and (12). Since the per period labour supplies are all constants, we can multiply them with the length of each respective phase and add the experience gained in former phases to find the labour experience at the end of each phase.

$$L_{t_2} = \left[\Psi - \frac{1}{w(L_{t_2}) + \alpha_1(t_2)\,[L_T - L_{t_2}]}\right](t_2 - t_1) \tag{15}$$

$$L_{t_3} = L_{t_2} + \left[\Psi - c(k, t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3)\,[L_T - L_{t_3}]}\right](t_3 - t_2) \tag{16}$$

$$L_T = L_{t_3} + \left[\Psi - \frac{1}{w(L_{t_3})}\right](T - t_3) \tag{17}$$

Jumps of Co-States

Proposition 1 *There is a discontinuous downward jump (upward) jump, when the return of labour experience is larger (smaller) during the first of the two phases. Furthermore one can show that the quotient of the two consecutive phases 1 and 2 is constant at t_2, when the experience derivative of income is constant within each phase.*

Proof. For $\alpha_i(t) \neq const$

$$\frac{\eta_1(t_2)}{\eta_2(t_2)} = \frac{\alpha_1(t_2)\left[L_T - L_{t_2}\right]}{w(L_{t_3}) - w(L_{t_2}) + \alpha_2(t_3)\left[L_T - L_{t_3}\right]} = \frac{\alpha_1(t_2)\left[L_T - L_{t_2}\right]}{\int_{t_2}^{t_3} \alpha_2(t) l_t dt + \alpha_2(t_3) \int_{t_3}^{T} L_t dt} \tag{18}$$

When the experience return is larger at a given point in time during phase 1 (in particular at t_2) than during phase 2, then the quotient (18) must be greater than one. Hence there is a downward jump of labour supply at t_2.

For $\alpha_i(t) = \alpha_i = const$

$$\frac{\eta_1(t_2)}{\eta_2(t_2)} = \frac{\alpha_1\left[L_T - L_{t_2}\right]}{w(L_{t_3}) - w(L_{t_2}) + \alpha_2\left[L_T - L_{t_3}\right]} = \frac{\alpha_1}{\alpha_2} \tag{19}$$

If $\alpha_2(t)$ decreases with time, then the denominator of (18) is smaller than that of (19), hence (18) must be larger than (19), which means that the upward jump is larger when $\alpha_i(t) \neq const$. ■

In order to simplify the continuative analysis, we assume that a_1 and α_2 are independent of time but $a_1 > \alpha_2$ as discussed earlier. The income payments at the end of phase one and two are then equal to the expressions,

$$w(L_{t_2}) = w(L_{t_1}) + \alpha_1 L_{t_2} \tag{20}$$

$$w(L_{t_3}) = w(L_{t_2}) + \alpha_2\left[L_{t_3} - L_{t_2}\right] \tag{21}$$

$$w(L_T) = w(L_{t_3}) \tag{22}$$

where $w(L_{t_1}) = w(0)$ is the income of an individual who has recently commenced working. (22) reminds us that there is no return on experience gained during phase 3. How results change, when we substitute $w(L_T) = w(L_{t_3}) + \alpha_3\left[L_T - L_{t_3}\right]$ for (22) where the experience return during phase 3 is $\alpha_3 \neq 0$, is briefly explored later.

2.2 The optimality condition for the timing of childbirth

There is the desire to have children earlier in life; and the probability that a child has a disability increases with the mother's age. This is modelled by a change in the expected cost. To keep things simple, we assume that $c(k, t_2)$ and $m(k, t_2)$ increase with certainty, when childbirth is delayed. Advanced medical research makes it feasible to give birth later in life, but such procedures are expensive. In addition to which, parents that are wealthier spend more money on raising their children. Since income increases in this model continuously, monetary costs $m(k, t_2)$ increase with t_2. Besides a positive derivative of $m(k, t_2)$ with respect to

t_2, we argue for a positive relation of time costs $c(k, t_2)$ and childbirth. The same rules that apply on the labour market also apply when people raise children: younger people can generally adapt better to changing market conditions and learn faster. A mother in her early 20s might be still able to drop off her children at the kindergarten, before going to her part-time job and pick them up again in the afternoon. Furthermore we assume that the length of time required to raise children is longer, when there are more children; $h'(k) > 0$. This term can be used later to evaluate policy implications for schools, where children can stay all day long. Once children are old enough to go to these schools, both parents could begin to work full-time again. In the model the individual then enters phase 3. We included monetary costs for phase 3 in an earlier working paper. Results shall be briefly discussed below.

With t_2 fixed, one can take the utility stream from $t_1$up to T and differentiate this expression with respect to t_2. This expression must be equal to zero at the optimal time of childbirth t_2^*. Now consider the following three sub-problems:

For $t \in [t_1, t_2]$ t_1 and t_2 fixed

$$SP_1^* = \max_{x_1(t)} \int_{t_1}^{t_2} \{x_1(t) + \ln[\Psi - l_1(t)]\} \, dt \tag{23}$$

$$s.t. \; \dot{l_1}(t) = l_1(t) \text{ and } w(L_t)l_1(t) - x_1(t) = 0$$

For $t \in [t_2, t_2 + h(k)]$ t_2 and $h(k)$ fixed

$$SP_2^* = \max_{x_2(t)} \int_{t_2}^{t_2+h(k)} \{x_2(t) + \ln[\Psi - l_2(t) - c(k, t_2)] + v_2(k)\} \, dt \tag{24}$$

$$s.t. \; \dot{l_2}(t) = l_2(t) \text{ and } w(L_t)l_2(t) - x_2(t) - m(k, t_2) = 0$$

For $t \in [t_2 + h(k), T] \quad t_2$, $h(k)$ and T fixed

$$SP_3^* = \max_{x_3(t)} \int_{t_2+h(k)}^{T} \{x_3(t) + \ln[\Psi - l_3(t)] + v_3(k)\}\, dt \tag{25}$$

$$s.t.\dot{l}_3(t) = l_3(t) \text{ and } w(L_t) l_3(t) - x_3(t) = 0$$

We need to use the Leibniz Rule to derive $\frac{\partial SP_i^*}{\partial t_2}$ for $i = 1, 2$ and 3. For each phase i we receive three terms:

1. The integral of $\frac{\partial SP_i^*}{\partial t_2}$ with the corresponding phase's bounds.
2. We subtract the t_2 derivative of the lower bound of phase i, which is multiplied by the Hamiltonian evaluated at the lower bound.
3. Finally we add the derivative of the upper bound with respect to t_2, which is multiplied by the Hamiltonian evaluated at that point.

Phase 1

$$\frac{\partial SP_1^*}{\partial t_2} = H_1^*(t_2) \tag{26}$$

Applying the envelope theorem, the first term is zero. The lower bound is independent of childbirth, hence term two is zero. The third term; $H_1^*(t_2)$ intuitively means that an incremental increase in t_2 comes along with additional per period utility gained during phase one at t_2.

Phase 2

$$\frac{\partial SP_2^*}{\partial t_2} = -\frac{\partial c(k, t_2)}{\partial t_2} \frac{h(k)}{\Psi - l_2^* - c(k, t_2)} - H_2^*(t_2) + H_2(t_3) \tag{27}$$

One can show that $H_2^*(t_3) - H_2^*(t_2) = 0$. This result is due to the fact that in the presence of learning, the per period utility within each phase is constant. The change in utility through an increase in consumption is completely offset by the change of utility through the decrease of the experience value. One can draw a parallel to the earlier discussion in section 2.1. Hamiltonians within any phase are of equal value independent of the period in which they are evaluated.

Applying the envelope theorem, the first term is $-\frac{\partial c(k,t_2)}{\partial t_2} \frac{h(k)}{\Psi - l_3^* - c(k,t_2)}$ and does not vanish here, because the derivative with respect to $c(k, t_2)$ is not equal to zero. However the derivatives of the per period Hamiltonian with respect to $x_2^*(t)$, l_2^* and $\eta_2^*(t)$, which have already been chosen optimally are zero. $c(k, t_2)$ depends on the number of children and the timing of childbirth, which are not optimal at this stage yet. The change of time costs has to be paid for the length of this phase, $h(k)$. The second term comes from a decrease of phase two's utility at the original t_2 before the change, the third term from an increase of phase two's utility at t_3. Phase two can be seen as shifted to the right within the time interval.

Phase 3

$$\frac{\partial SP_3^*}{\partial t_2} = -H_3^*\left(t_3\right) \tag{28}$$

The envelope theorem allows the first term to vanish, the third term does not occur here either, because the upper bound of phase four T is exogenously given and hence independent of t_2. $-H_3^*\left(t_3\right)$ expresses the fact that phase three becomes shorter and looses an incremental period at t_3.

Adding (26), (27) and (28) and setting them equal to zero gives the optimality condition for the optimal timing of childbirth, where k is still assumed to be fixed.

$$H_1^*\left(t_2\right) - \frac{\partial c(k,t_2)}{\partial t_2}\frac{h(k)}{\Psi - l_3^* - c(k,t_2)} - H_3^*\left(t_3\right) \doteq 0 \tag{29}$$

2.3 The optimality condition for the number of children

Again we use the Leibniz rule and the Envelope theorem with the same method used to derive the t_2^*-optimality condition. The timing of childbirth depends on phase one's utility stream, but the number of children k does not, thus $\frac{\partial SP_1^*}{\partial k} = 0$.

The length of phase two and three changes with the number of children. The terms that affect the number of children are the costs and benefits, while children are young (phase 2), the benefits when are older (phase 3), and the length of phase 2, $h(k)$.

Phase 2

$$\frac{\partial SP_2^*}{\partial k} = \left[\frac{\partial v_2(k)}{\partial k} - \frac{\partial c(k,t_2)}{\partial k}\frac{1}{\Psi - l_2^* - c(k,t_2)}\right]h(k) + H_2^*(t_3)h'(k) \tag{30}$$

The first term is the change of the per period utility of phase two from an increase of benefits from having more children, subtracted by additional costs multiplied by the length of this phase $h(k)$. The second term is the additional utility from an increase of length of phase two.

Phase 3

$$\frac{\partial SP_3^*}{\partial k} = (T - t_3)\frac{\partial v_3(k)}{\partial k} - h'(k)H_3^*\left(t_3\right) \tag{31}$$

When more children are born, the additional benefit from having them is accounted for by the first term. Phase 3 becomes shorter through an increase of length in phase 2 when more children are present (second term) .

The k^*-optimality condition is thus given by

$$h(k)\left[\frac{\partial v_2(k)}{\partial k}-\frac{\partial c(k,t_2)}{\partial k}\frac{1}{\Psi-l_2^*-c(k,t_2)}\right]+(T-t_3)\frac{\partial v_3(k)}{\partial k}+h'(k)\left[H_2^*(t_3)-H_3^*(t_3)\right]\doteq 0 \tag{32}$$

We derive the optimal number of children and the optimal timing of childbirth simultaneously. The equation that describes the optimal number of children is given by (32), which depends on t_2 just in the same way as (29), the equation that characterizes the optimal date of childbirth.

Given (15), (16), (17), (20), (21), (22), (29) and (32) we can solve for the optimal number of children and timing of childbirth numerically. Besides these two variables, we can also solve for cumulative labour experience at t_2, t_3, and T and the per period income level at these points. The characterization of an analytical solution would be extremely tedious, because one would have to apply the implicit function theorem for eight equations, where each of them depends on all other seven equations.

2.4 Results

We need to make assumptions regarding the functional forms of the cost functions, utility derived from children and the length of phase two. These are presented in figure (2).[4]

All functions in figure (2) are convave in the number of children k. When they depend on the timing of childbirth, then they are convex in t_2. The parameters have also been choosen such that the optimal number of children is 2 .2 to reflect the number of children a woman must have on average to keep the population at a constant level. In 2006, the average age of a woman receiving her first child in the 25 European Union member states was approximately 29 years of age.[5] The parameters of the baseline model are chosen to have an optimal number of years spent on the labour market of about 7.4 years, because an average age, when entering the labour market of 21.6 seems reasonable.[6] T, the total number of years spent on the labour market is assumed to be 40. The age at retirement is thus 61.6. The parameters α_1 and α_2 are 5% and 2%, reflecting the

[4] We use Matlab to find numerical solutions for the eight conditions; the command "fsolve" finds solutions for nonlinear systems.

[5] Eurostat (2006): Population statistics

[6] Within the EU-15 countries over 40% of the cohord aged 22 years has entered the labour force.

	Functional Form	*Phase*
Time costs	$c(k,t_2) = c_1 k^{\beta_1} + c_2^{-1} t_2^{\beta_2}$	2
Utility from children	$v_2(k) = c_3 k^{\beta_3}$	2
Utility from children	$v_3(k) = c_4 k^{\beta_3}$	3
Time length of raising children	$h(k) = c_5 k^{\beta_4}$	2
Monetary costs	$m(k,t_2) = c_6 k^{\beta_5} + c_2^{-1} t_2^{\beta_2}$	2

Figure 2: Functional forms of cost and utility functions from having children and the length of phase 2.

Values of Baseline Model

t_2^*	7.447	L_{t2}^*	74.200	$w(t_2)^*$	13.710
k^*	2.189	L_{t3}^*	95.806	$w(t_3)^*$	14.142
		L_T^*	345.583	$w(T)^*$	14.142

How a 1% increase of parameters below affects t_2 and k

Variables	w_{t1}=10	α_1=0.05	α_2=0.02	t_1=0 (+0.1)	T=40
t_2^*	7.5035	7.7503	7.4299	7.4192	7.615
k^*	2.1604	2.1147	2.1901	2.203	2.1829
	c_1=4	c_2=50	c_3=70	c_4=70	c_5=5
t_2^*	7.5908	7.5228	7.3385	7.1978	7.4841
k^*	2.1307	2.1742	2.2309	2.2565	2.1521
	c_6=20	β_1=0.05	β_2=2	β_3=0.03	β_4=0.03
t_2^*	7.474	7.5896	7.0766	7.1819	7.5548
k^*	2.1785	2.1333	2.2636	2.2854	2.1392
	β_5=0.05	Ψ=10			
t_2^*	7.4619	8.0552			
k^*	2.1831	2.0429			

Figure 3: How the optimal number of children and the timing is affected by the underlying parameters

observation that income increases with experience more during phase 1 when no children are present and less when she works part-time and looks after her children (phase 2). Empirically one does not observe an increase of real income during phase 3, hence we set $\alpha_3 = 0$. We start at an exogenously given wage of 10. It endogenously increases to 13.7 until t_2^*, furthermore goes up to 14.1 during phase 2 and remains at this level until T. Comparative static results are summarized in figure (3). To save space we left out how other variables such as labour experience and the wage rate are affected through a parameter change. Green (red) values represent increasing (decreasing) t_2^*'s or k^*'s due to a 1% increasing parameter.

Changing one of the underlying parameters affects all optimality conditions. A first observation is that when k^* increases (decreases) due to a change of one parameter, then the timing of childbirth t_2^* decreases (increases). Besides the negative correlation between these variables, there is a negative correlation between k^* and all other variables; the optimal number of children increases only, when the optimal cumulative labour supplies and incomes at the end of all phases decrease. We interpret the results one for one and concentrate on the timing of childbirth and the number of children. An increase in the income level decreases the number of children wanted. The opportunity costs of having children increases, thus less children are born. An increase in α_1 delays the optimal timing of childbirth, because an individual wants to exploit income increases during phase 1, which are larger than in any other phase. A delayed timing of childbirth is automatically connected to fewer children. An increase in α_2 on the other hand increases the number of children wanted, because an early childbirth is not as expensive, when her wage can still increase sufficiently after t_2^*. In an earlier version, we accounted for $\alpha_3 > 0$; labour experience gained during phase 3 increases the future income. Increasing α_3 has the same comparative effects on the choice variables as increasing α_2 with the same intuition behind it.

An increasing working-span of an individual (changes in t_1, T and Ψ) has a negative effect on fertility. An increase of the working life raises life-time income and income per period. Thus the opportunity costs of having children are larger. An increase of c_1 or β_1 means that the marginal time cost of an additional child increases. Not surprisingly, if these costs increase, the number of children goes down. Governments that offer placements in kindergartens, where children can stay until the afternoon, give the mother the opportunity to take a longer part-time job and hence decrease c_1. c_2 and β_2 are parameters that are connected to the time cost burden of raising children, when children come late. Up to a number of $\sqrt{c_2}$ years, the time costs reflected by the second term of $c(k, t_2)$ are less than one. Since they increase exponentially though, they do matter at some point and induce her to enter phase 2. When c_2 increases or β_2 decreases, the marginal time cost of giving birth late decreases. Therefore women have fewer children but later. Medical research enabling late childbirth has a negative effect on fertility. Soares (2005) shows why advances in medical research corresponds with lower fertility in developing countries. When child mortality is reduced, the expected costs of large families increase and the marginal benefits decrease. An increase of benefits from young and old children c_3, c_4 and β_3 increase the number of children. If the length of phase 2 is long (large c_5 and β_4), then the individual's number of children decreases. The results come from the underlying structure of the model based on costs mainly occuring due to leaving phase 1 and entering phase 2 (decrease of cumulative experience return), but benefits also occur during the last phase. A government that offers sufficient placements of full-day care centres or full-time schools increases its country's fertility, by shortening phase 2. β_5 and c_6 are connected to the monetary costs she has to encounter, when children are young. An increase of child benefits increases the number of children. It is straightforward to include monetary costs for phase 3 as well. Changing the parameters of these, when they have the same functional form as $m(k, t_2)$ also has the same effect as changing β_5 and c_6. Child benefits are reflected by a lower c_6. A financial incentive given to parents in Germany is the so-called "Elterngeld" (parental benefits). Parents receive up to 2/3 of one of the partner's last net income for one year, if one parent stays at home during that time and looks after the child. Parents can choose between a one-year-parental-leave and a day-care centre. In our model this would be reflected by the choice between a positive c_6 and a lower $c(k, t_2)$ if the parental leave is rejected and a negative c_6 and a very large $c(k, t_2)$ such that $l_2^* = 0$ if it is accepted. Apps and Rees (2004) also show how specific government policies affect fertility choices.

3 Extension A: Divorce

Marriage may not last until the end of a woman's planning horizon T. [7] When the probability of divorce increases through an exogenous change, then

Grossbard-Shechtman (1984) argues that women have more outside options and reduce their supply of household goods which includes the number of children. In our set-up divorce causes the number of children to be reduced as well, but for a different reason. Divorce is more costly for a woman when she has more children. A woman with many children has less labour experience and hence a lower income. At the same time the costs of having children increase after divorce, because she has to raise them by herself $c^d(k, t_2) > c(k, t_2)$ and receives less monetary support from the father, hence her monetary contribution to children increases $m^d(k, t_2) > m(k, t_2)$. Benefits from children for the mother do not change after divorce. A divorce soley effects the woman's utility, when it occurs during phase 2, therefore we also restrict it to that phase.

Phase 2 $t \in [t_2, t_3]$ is solved in two steps

1. The date of divorce d is known and $d \in [t_2, t_3]$
2. The date of divorce is uncertain.

3.1 Step 1: The Optimal Plan before and after Divorce known to occur at time d.

Optimal Plan after d We begin to solve the problem by finding the individual's optimal plan after divorce has occurred. Later it is shown, how the

individual acts before the known date d.

The objective that needs to be maximized is

[7] Sweden and the United Kingdom have the highest divorce rates in Europe with over 50%. Austria, Belgium, Czech Republic, Denmark, Finland, France, Germany, Hungary, Norway and Switzerland have divorce rates between 40%-50%. Ireland, Italy, Poland and Spain have the lowest divorce rates of less than 20% according to Eurostat (2006) "Population Statistics".

$V_2^d = \int_d^{t_3} \left\{ x_2(t) + \ln \left[\Psi - l_2(t) - c^d(k, t_2) \right] + v_2(k) \right\} dt + V_3^*$ subject to the budget constraint $w(L_t) l_2(t) = x_2(t) + m^d(k, t_2)$ and as before $\dot{L}_t = l_2(t)$. The Lagrangian after divorce is

$$\begin{aligned} \Gamma_2^d \left[x_2(t), l_2(t) \right] &= x_2(t) + \ln \left[\Psi - l_2(t) - c^d(k, t_2) \right] + v_2(k) \\ &\quad + \eta_2(t) l_2(t) + \lambda_2(t) \left[w(L_t) l_2(t) - x_2(t) - m^d(k, t_2) \right] \end{aligned} \tag{33}$$

Substituting $\lambda_2(t) = 1$ the equilibrium conditions of this problem are

$$l_2^d(t) = \Psi - c^d(k, t_2) - \frac{1}{\eta_2(t) + w(L_t)} \tag{34}$$

$$\dot{\eta}_2(t) = -\dot{w}(L_t) \tag{35}$$

The transversality condition here is an expression of the co-state at t_3. Working an additional hour at t_3 increases her income and has a future return of

$$\eta_2^d(t_3) = \frac{\partial w(L_t)}{\partial L_t} |_{t=t_3} \int_{t_3}^{T} L_T dt = \frac{\partial w(L_t)}{\partial L_t} |_{t=t_3} \left[L_T - L_{t_3} \right] \tag{36}$$

Given transversality condition (36), equation (35) can be re-written such that the co-state becomes

$$\eta_2^d(t) = w(L_{t_3}) - w(L_t) + \alpha_2(t_3) \left[L_T - L_{t_3} \right] \tag{37}$$

where $\alpha_2(t_3) = \frac{\partial w_3(L_t)}{\partial L_t} |_{t=t_3}$.

Using (34) and (37) we solve for the optimal labour supply, which is independent of time and its consumption counterpart, which does depend on time just as in the absence of divorce,

$$l_2^d = \Psi - c^d(k, t_2) - \frac{1}{w(L_{t_3}) + \alpha_2(t_3) \left[L_T - L_{t_3} \right]} \tag{38}$$

$$x_2^d(t) = \Psi w(L_t) - c^d(k, t_2) w(L_t) - \frac{w(L_t)}{w(L_{t_3}) + \alpha_2(t_3) \left[L_T - L_{t_3} \right]} - m^d(k, t_2) \tag{39}$$

The direct utility after divorce $V_2^d(d)$, which is needed to find the optimal number of children later is

$$V_2^d(d) = \int_d^{t_3} \left\{ x_2^d(t) + \ln \left[\Psi - l_2^d - c^d(k, t_2) \right] + v_2(k) \right\} dt + V_3^* \tag{40}$$

and the per-period direct utility, needed for the same reason, is

$$V_2^d(t) = \begin{array}{c} \Psi w(L_t) - c^d(k, t_2) w(L_t) - \frac{w(L_t)}{w(L_{t_3}) + \alpha_2(t_3) \left[L_T - L_{t_3} \right]} - m^d(k, t_2) \\ + \ln \left[\frac{1}{w(L_{t_3}) + \alpha_2(t_3) \left[L_T - L_{t_3} \right]} \right] + v_2(k) \end{array} \tag{41}$$

Optimal Plan before d We solve for an optimal plan for a known date of divorce d. The individual maximizes the objective $\int_{t_2}^{d}\{x_2(t)+\ln[\Psi-l_2(t)-c(k,t_2)]+v_2(k)\}\,dt$ subject to the constraint $w(L_t)l_2(t)=x_2(t)+m(k,t_2)$.

$$\dot{\eta_2}(t)=-\dot{w}(L_t) \tag{42}$$

together with the transversality condition

$$\eta_2(d)=\alpha_2(d)\int_d^T l(t)dt \tag{43}$$

yields the co-state's functional equation

$$\eta_2(t)=w(L(d))-w(L_t)+\alpha_2(d)\left[L_T-L(d)\right] \tag{44}$$

Proposition 2 *The co-state function does not change after a known date of divorce, when the experience derivative of income is time independent $\alpha_2(t)=\alpha_2$.*

Proof. Substituting α_2 for $\alpha_2(t)$ in (37)

$$\begin{aligned}
\eta_2^d(t) &= w(L_{t_3})-w(L_t)+\alpha_2\left[L_T-L_{t_3}\right]\\
&= w(L(d))+\int_d^{t_3}\dot{w}(L_t)dt-w(L_t)+\alpha_2\left[L_T-L_{t_3}\right]\\
&= w(L(d))+\int_d^{t_3}\alpha_2 l(t)dt-w(L_t)+\alpha_2\left[L_T-L_{t_3}\right]\\
&= w(L(d))+\alpha_2\left[L_{t_3}-L(d)\right]-w(L_t)+\alpha_2\left[L_T-L_{t_3}\right]
\end{aligned}$$

which is equal to (44). ■

The result here is also due to the utility's functional form. If it were not quasi-linear in the consumption good, then $\lambda_2(t)\neq 1$ and the co-state would depend on per-period labour or consumption.

$$l_2=\Psi-c(k,t_2)-\frac{1}{w(L_{t_3})+\alpha_2(t_3)\left[L_T-L_{t_3}\right]} \tag{45}$$

$$x_2(t)=\Psi w(L_t)-c(k,t_2)w(L_t)-\frac{w(L_t)}{w(L_{t_3})+\alpha_2(t_3)\left[L_T-L_{t_3}\right]}-m(k,t_2) \tag{46}$$

(46) shows that consumption is larger before than after divorce has occurred, because $c(k,t_2)<c^d(k,t_2)$ and $m(k,t_2)<m^d(k,t_2)$.

$l_2>l_2^d$ because children demand more time for their child care. Future benefits of labour remain unchanged.

3.2 Step 2: The optimal plan before an unknown date of divorce

Decisions after d are given in the last section; see (38) and (39). They do not vary, when divorce is uncertain, because after d all uncertainty is cleared. Expectations about divorce are uniform among all representatives, the subjective probability that divorce occurs at time t is $\phi(t)$. The perceived probability that the marriage will persist at least until time t is consequently calculated as

$$G(t) = \int_{t}^{t_3} \phi(t)dt \tag{47}$$

The date of divorce is unknown; the individual is obliged to maximize her expected utility,

$$\int_{t_2}^{t_3} \phi(d)H(d)dd + \int_{t_2}^{t_3} \phi(d)V_2^d(d)dd \tag{48}$$

where $V_2^d(d)$ is given by (40) and $H(d) = \int_{t_2}^{d} u_2\left[x_2(t), l_2\right] dt$ $= \int_{t_2}^{d} \left\{x_2(t) + \ln\left[\Psi - l_2(t) - c(k, t_2)\right] + v_2(k)\right\} dt$. (48), upon integration by parts may be expressed as

$$\int_{t_2}^{t_3} \left\{G(t)u_2\left[x_2(t), l_2\right] + \phi(t)V_2^d(t)\right\} dt \tag{49}$$

where $V_2^d(t)$ is given by (41).
Therefore an individual maximizes

$$\int_{t_2}^{t_3} \left\{G(t)u_2\left[x_2(t), l_2\right] + \phi(t)V_2^d(t)\right\} dt + V_3^* \tag{50}$$

subject to the known constraints. Consequently, the Lagrangian from which the socially optimal plan before divorce can be derived is

$$\begin{aligned} \Gamma = {} & G(t)\left\{x_2(t) + \ln\left[\Psi - l_2(t) - c(k, t_2)\right] + v_2(k)\right\} + \phi(t)V_2^d(L_t) \\ & + \eta_2^{bd}(t)l_2(t) + \lambda_2(t)\left[w(L_t)l_2(t) - x_2(t) - m(k, t_2)\right] \end{aligned} \tag{51}$$

where the equilibrium conditions are

$$\lambda_2(t) = G(t) \tag{52}$$

$$l_2(t) = \Psi - c(k, t_2) - \frac{G(t)}{\eta^{bd}(t) + G(t)w(L_t)} \tag{53}$$

$$\dot{\eta_2^{bd}}(t) = -\phi(t)\frac{\partial V_2^d}{\partial L_t} - w(L_t)G(t) \tag{54}$$

where $\eta_2^{bd}(t)$ is the co-state of phase 3 before divorce, when divorce is uncertain.

The per period consumption is

$$x_2(t) = \Psi w(L_t) - c(k, t_2)w(L_t) - \frac{G(t)w(L_t)}{\eta_2^{bd}(t) + \lambda_2(t)w(L_t)} - m(k, t_2) \tag{55}$$

The expected direct utility in the presence of uncertainty (index U) at t_2 for all future periods of phase 3 is

$$V_2^U(k, L_t) = \int_{t_2}^{t_3} \left\{G(t)u_2\left[x_2^*(t), l_2^*\right] + \phi(t)V_2^d(L_t)\right\} dt + V_3^* \tag{56}$$

The co-state's time derivatives before and after divorce in the absence of uncertainty (42) and (35) respectively denoted by $\dot{\eta_2}(t)$ are equal. Comparing these with (54) denoted by $\dot{\eta_2^{bd}}(t)$ indicates the timing of childbirth, when divorce is uncertain. Both equations are used to derive

$$\dot{\eta_2}(t) = \frac{\dot{\eta_2^{bd}}(t) + \phi(t)\frac{\partial V_2^d}{\partial L_t}}{G(t)} \tag{57}$$

The co-states' time derivative and therefore also the co-states themselves are equal, when the probability of divorce at some time t, $\phi(t) = 0$, and the perceived probability that marriage will persist at least until time t, $G(t) = 1$. The second term in the nominator of (57) is small, because the instantaneous probability of divorce $\phi(t)$ is small. $G(t)$ is the probability that a couple is still married at time t. In most EU countries except the UK and Sweden this value is at least 0.5 for all $t \in [t_2, t_3]$. Thus one can assume that $G(t) > \phi(t)\frac{\partial V_2^d}{\partial L_t}$. Both time derivatives are negative, because within this phase and any other phase, experience pays off less and less the sooner she reaches her retirement, therefore $\dot{\eta_2^{bd}}(t) < \dot{\eta_2}(t)$. Both co-state functions have the same functional value at t_3, because all uncertainty is resolved at t_3. In the case of no divorce $\eta_2^{bd}(t)$ must lie entirely above $\eta_2(t)$. They coincide at t_3. In case a divorce occurs, $\eta_2^{bd}(t)$ must jump downwards such that both co-states can coincide. This is shown in figure (4).

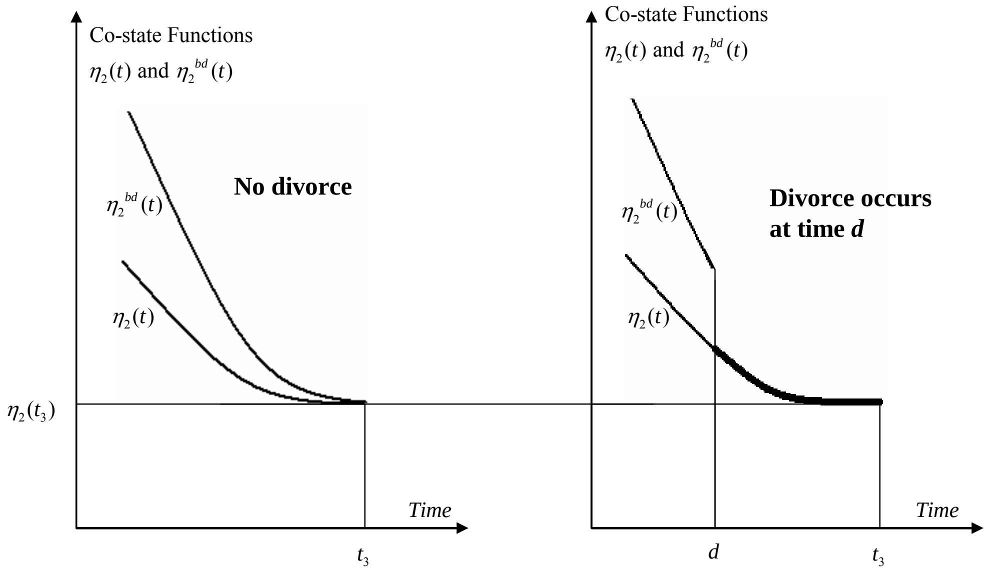

Figure 4: The co-states of phase two, when a divorce does not occur and when it does for a known and an unknown d.

When the date of divorce is known, then the co-state before and after divorce is unchanged. It is only affected, when d is unknown. This shows that our individual values labour more, when she faces the risk of divorce. She therefore has a higher labour supply in the presence of uncertainty. A known date of divorce would therefore lead to a lower labour supply and more children due to the negative correlation between these variables. Next we answer the question whether a woman reduces the number of children in the presence of divorce and if she consequently delays the timing of childbirth.

4 Extension B: Divorce, a numerical simulation

After illustrating divorce within this model set-up analytically such that there is a positive probability of divorce in every period of phase 2 (extension A), we continue to show a simplified method where divorce occurs with a positive probability at varying points in time between t_2 and t_3. Derivations from extension A are needed in this section. The divorce probability is zero for all other periods as in extension A, because a woman would not be affected by it in this set-up. Again it's a straightforward extension to include divorce for phase 3, when there are monetary costs connected to children in that phase. Our main results do not change, hence we leave it out, however we discuss them briefly below. Within this framework, we can solve for the timing of fertility and the number of children numerically as we have done in the baseline model. Extension A was more general therefore less precise, because it only characterizes the co-state during phase 2 in the presence of divorce, but does not find a solution for t_2^* and k^* explicitly, which this section does. With a probability of $p < 1$

there is a divorce during phase 2. Re-marriages are excluded for simplicity. The possible date of divorce d during phase 2 is given by

$$d = t_2 + \frac{h(k)}{c_7} \tag{58}$$

where $c_7 \in (1, \infty)$. (58) means that divorce occurs after a certain portion of phase 3 is over, which depends on c_7. The longer phase 3 the more children are present; $h'(k) > 0$. Divorce occurs then later as it is more costly, when more children are present. Next we derive the t_2 and k- optimality conditions. Again we differentiate utility streams. The first and third phases' utilities do not change through divorce but their utility stream needs to be added to the two cases: divorce and no-divorce. The utility streams from t_1 to T are thus;

1. No divorce: (23)+(24)+(25)
2. Divorce during phase 2: (23)+$DP_{2.1}^* + DP_{2.2d}^*$+(25).[8]

For $t \in [t_2, d]$, t_2 and d fixed

$$DP_{2.1}^* = \max_{x_2(t)} \int_{t_2}^{d} \{x_2(t) + \ln[\Psi - l_2(t) - c(k, t_2)] + v_2(k)\}\, dt \tag{59}$$

$$s.t.\ \dot{l_2}(t) = l_2(t) \text{ and } w(L_t)l_2(t) - x_2(t) - m(k, t_2) = 0$$

For $t \in [d, t_2 + h(k)]$ t_2, $h(k)$ and d fixed

$$DP_{2.2d}^* = \max_{x_2(t)} \int_{d}^{t_2+h(k)} \left\{x_2(t) + \ln\left[\Psi - l_2(t) - c^d(k, t_2)\right] + v_2(k)\right\} dt \tag{60}$$

$$s.t.\ \dot{l_2}(t) = l_2(t) \text{ and } w(L_t)l_2(t) - x_2(t) - m^d(k, t_2) = 0$$

[8]The subscript 2.1 is attached to the utility during phase 2 up to d and 2.2d to the utility during phase 2 after d.

Case 1, the no divorce case is described by the baseline model. The left hand side of (29) multiplied by the no-divorce probability is the first part of the expected utility. Case 2: We have already solved for $DP^*_{2.1}+DP^*_{2.2d}$ in extension A. The t_2-optimality conditions can be derived when adding the terms of our two cases:

1. The expected utility from "no-divorce" case for the t_2-optimality condition is given by
$$(1-p)\left[H_2^*(t_2) - \frac{\partial c(k,t_2)}{\partial t_2}\frac{h(k)}{\Psi - l_2^* - c(k,t_2)} - H_3^*(t_3)\right] \tag{61}$$
2. The part, when divorce occurs at d during phase 3 is
$$p_3\left\{\begin{array}{c} H_2^*(t_2) - H_2^{d*}(t_2+h(k)) - \frac{1}{\Psi - l_2^* - c(t_2,k)} * \\ \left[\frac{h(k)}{c_7}\frac{\partial c(k,t_2)}{\partial t_2} + h(k)\left(1-c_7^{-1}\right)\frac{\partial c^d(k,t_2)}{\partial t_2}\right] \end{array}\right\} \tag{62}$$
.
The quotient $\frac{1}{\Psi - l_2^* - c(t_2,k)}$ is equal after and before divorce, because the change of the labour supply and the change of the children's time costs $c(t_2,k)$ cancel. For the not-divorce and for the divorce case, hamiltonians of the same phase evaluated at different periods are equal such that $H_2^*(t_2+\frac{h(k)}{c_7}) - H_2^*(t_2) = 0$ and $H_2^{d*}(t_2+h(k)) - H_2^{d*}(t_2+\frac{h(k)}{c_7}) = 0$.

Adding (61) and (62) and setting these terms equal to zero is the t_2-optimality condition, when divorce is a possibility within a marriage.

The k-optimality condition is derived next.

1. Case 1: the probability of no-divorce is multiplied with the LHS of equation (32);
$$(1-p)\left\{\begin{array}{c} h(k)\left[\frac{\partial v_2(k)}{\partial k} - \frac{\partial c(k,t_2)}{\partial k}\frac{1}{\Psi - l_2^* - c(k,t_2)}\right] \\ +(T-t_3)\frac{\partial v_3(k)}{\partial k} + h'(k)\left[H_2^*(t_3) - H_3^*(t_3)\right] \end{array}\right\} \tag{63}$$
2. Case 2: divorce at d:
$$p\left\{\begin{array}{c} h(k)\left[\frac{\partial v_2(k)}{\partial k} - \frac{\partial c(k,t_2)}{\partial k}\frac{1}{\Psi - l_2^* - c(k,t_2)}\right] + (T-t_3)\frac{\partial v_3(k)}{\partial k} \\ +\frac{h'(k)}{c_7}\left[H_2^*(d) - H_2^{d*}(d)\right] + h'(k)\left[H_2^{d*}(t_3) - H_3^{d*}(t_3)\right] \end{array}\right\} \tag{64}$$
The first two terms are the same as in case 1. The length of phase 2 increases with k by $h'(k)$, also remember that d is positively dependent on $h(k)$. When divorce occurs at d, then the first part of phase 2 [t_2, d_3] increases, because divorce occurs later (term 3). At the same time phase 2 becomes longer and phase 3 becomes shorter (term 4).

		Parameter increases by 1%			
Baseline	*Effect on variables*	c_7=2	c_8=2	c_9=1.5	p=0.4
75.6536	L_{t2}	75.6980	75.6994	75.7193	75.6926
98.6950	L_{t3}	98.7433	98.7523	98.7553	98.7389
354.0908	L_T	354.1627	354.2116	354.1273	354.1630
13.7827	w_{t2}	13.7849	13.7850	13.7860	13.7846
14.2435	w_{t3}	14.2458	14.2460	14.2467	14.2456
14.2435	w_T	14.2458	14.2460	14.2467	14.2456
7.5928	t_2	7.5972	7.5974	7.5994	7.5967
1.7887	k	1.7850	1.7828	1.7865	1.7851

Figure 5: The effect of divorce related parameters on the variables of the model

Setting the sum of (63) and (64) equal to zero, is the k- optimality condition in the presence of divorce. We can continue with the numerical simulation to find k^* and t_2^*. We assume that a woman's time costs which occur during phase

2, when raising children increase to $c_8 c(k, t_2)$ after she had a divorce. Monetary costs during phase 2 change to $c_9 m(k, t_2)$. The parameters c_8 and c_9 must all be larger than one. Values of newly introduced parameters, where p is the divorce probability and c_7 the timing when divorce occurs within this phase are given in the second line of figure 5. The probability of divorce is assumed to be 40%. Divorce occurs half way through within each phase, time costs are doubled and monetary costs increase by one half. All other parameters are the same as in the baseline model. The results for divorce are summarized by figure 5.

The number of children in the presence of divorce decreases to $k^* = 1.79$ from around 2.2 in the baseline model, where divorce was excluded from the analysis. $k^* = 1.79$ is closer to the average of the number of children a woman within the European Union countries gave birth to in 2007. In 2007 the fertility rate within the 27 European member states was between 1.25 (Slovakia) and 1.98 (France).[9] Not surprisingly, an increase of all divorce related parameters delays childbirth and yields a decrease of the optimal number of children. The burden of divorce is largest when children are young. We extended this present model to allow for divoce during phase 3; children would not need to be looked after, however they still receive a monetary transfer from their parents: $c_3(k, t_2) = 0$ and $m_3(k, t_2) > 0$. A change of the divorce probability of phase 3 affects fertility

[9] European Commission, Eurostat: Statistics in focus 81/2008, Population and social conditions.

less than a change of the divorce probability of phase 2, because costs are larger during phase 2, when time is devoted to rasing children c_8.

5 Conclusions

This model has been the first to solve simultaneously for the optimal timing of childbirth and number of children in a continuous time framework, where the wage is determined by work experience in a way that depends on the life phase in which it is accumulated. It shows that the date of childbirth and the number of children are negatively related. The marginal value of labour jumps when labour experience influences income differently, which is most likely to be the case when one changes from a full-time to a part-time job. A steep income profile right after leaving school has a negative effect on fertility, while a steep income profile when raising children and afterwards affects fertility positively.

We have shown the effects of the different types of cost of raising children, time costs and money costs. Individuals with high returns from education spend more time in education and have fewer children. Women value market work more when they face the risk of divorce, and so fertility is delayed and fewer children are born. The largest impact of divorce is when the probability of divorce during the phase in which the children are at home is large. Then a woman has to bear larger monetary costs, but even more importantly she has to devote more of her time towards child care. This has two negative effects: her current and future income decrease, because she is forced to work less on the labour market. Overall, the results of our model appear to be consistent with what empirical evidence is available on these relationships.

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7 Appendix

If one includes an education phase in the analysis and does not just assume an exogenous initial income, then one can derive fertility results based on ability θ. During the first phase $t \in [0, t_1]$, a woman invests in her education. The income

level $w_1(t_1) = \theta\sqrt{t_1}$ at which the individual starts working depends on the length of time spent at school t_1 and the ability θ. We assume that education is free. For simplicity the individual receives a constant amount of income each period s, which translates into a per period utility of $u_1(s) = s$. Given the optimal labour supplies for all phases, the initial income level, experience returns and the points in time t_1, t_2, t_3 and T, we can compute the life time utility

$$V_0^* = \int_0^{t_1} s dt + V_1^* \tag{65}$$

where V_1^* is the utility stream of phases one, two and three from the baseline model. When t_1 is given, then the first term of (65) becomes st_1 and it is just an additive constant and does not play a role for the derivation of the number of children and the timing of childbirth. The utility streams of all other phases do matter, when one derives t_2 and the optimal number of children k. We endogenize the number of years being educated and the initial income receivd on the labour market t_1 and $w(t_1)$. The larger somebody's ability, the larger is her income level at t_1 , when the person starts working. Ability is independent from any further income increases through experience, but it affects the income

Baseline	Effect on variables	*Parameter increases by 1%* e=3	s=10	β_5=0.09	θ=10
37.888	L_{t2}	37.445	37.916	39.781	38.890
59.503	L_{t3}	59.067	59.530	61.281	60.460
308.280	L_T	307.868	308.301	309.677	309.068
15.529	w_{t2}	15.489	15.539	16.134	15.803
15.962	w_{t3}	15.921	15.971	16.564	16.234
15.962	w_T	15.921	15.971	16.564	16.234
8.213	t_2	8.197	8.217	8.456	8.324
1.824	k	1.831	1.822	1.718	1.775
13.635	w_{t1}	13.616	13.643	14.145	13.859
4.411	t_1	4.439	4.412	4.465	4.421

Figure 6: How the education related parameters influence the underlying variables of the model

level during all future periods. A function that fulfills these requirements is (66) .

$$w(L_{t_1}) = \theta\,(t_1 - e)^{\beta_5} \tag{66}$$

where $0 < \beta_5 < 1$ and e is minimum number of years being educated that is required to obtain a degree. The second equation is the optimality condition for t_1^* (67).

$$(T - t_1)\frac{\partial w(t_1)}{\partial t_1} + \frac{\partial V_1}{\partial t_1} \doteq 0 \tag{67}$$

The first term is the benefit from an increase of the income level from an additional year in school, the second term is the change of future utility from loosing one year, which is equal to $s - H_1^*(t_1)$. (66) and (67), together with the 8 equations from the baseline model solve for the optimal timing of entering the labour market and the initial income simultaneously. We use the same parameters that were used in baseline model (figure 3), additional parameters are given in figure (6). The results from the numerical simulation for the baseline case are given in the left column of figure (6). After $t_1^* = 4.411$ years of being educated, a female enters the job market. Her wage is then 13.635€ and after about 3.8 years of working full-time $(t_2 - t_1)$ she begins to work part-time and has $k^* = 1.824$ children. Figure (6) also shows the effects on all variables, when we increase e, s, β_5, and θ by one per cent. Green values are associated with increases, while red values present decreases.

Again k and t_2 move in opposite directions as in the baseline model. The high ability types (high θ and β_5) have later fewer children, because they were educated longer and have a higher initial income; the opportunity costs of raising children are larger. Receiving more financial support, while being in school (high s) delays the timing of childbirth and increases the income level, hence the number of children decreases. When the minimum number of years needed to graduate e increases, more time is spent at school, therefore total life time income cannot rise by as much. Opportunity costs of raising children decreases and more children are born. De la Croix and Doepke (2003) show that less educated families decide to have more children, in whose education they invest little. When inequality within a society is high, large fertility differentials lower the growth rate of average human capital, because poor families that spend less on education make up a large proportion of the next generation. The fertility differential is a function of the income distribution. It increases with inequality; therefore countries with higher inequality accumulate less human capital, and therefore, grow slower.